Studentized processes of U-statistics

Masoud M. Nasari*

School of Mathematics and Statistics, Carleton University,

1125 Colonel By Drive, Ottawa, Canada

e-mail: mmnasari@connect.carleton.ca

Abstract

A uniform in probability approximation is established for Studentized processes of non degenerate U-statistics of order $m \geq 2$ in terms of a standard Wiener process. The classical condition that the second moment of kernel of the underlying U-statistic exists is relaxed to having $\frac{5}{3}$ moments. Furthermore, the conditional expectation of the kernel is only assumed to be in the domain of attraction of the normal law (instead of the classical two moment condition).

1 Introduction and Background

Let $X_1, X_2, \ldots,$ be a sequence of non-degenerate real-valued i.i.d. random variables with distribution F. Let $h(X_1, \ldots, X_m)$, symmetric in its arguments, be a Borel-measurable real-valued kernel of order $m \geq 1$, and consider the parameter $\theta = \int_{\mathbb{R}^m} h(x_1, \ldots, x_m)\, dF(x_1) \ldots dF(x_m) < \infty$. The corresponding U-statistic (cf. Serfling [?] or Hoeffding [?]) is

$$U_n = nm^{-1} \sum_{C(n,m)} h(X_{i_1}, \ldots, X_{i_m}) = \ [n]^{-m} \sum_{C'(n,m)} h(X_{i_1}, \ldots, X_{i_m}),$$

where $m \leq n$, $\sum_{C(n,m)}$ and $\sum_{C'(n,m)}$ respectively stand for summing over $C(n,m) = \{1 \leq i_1 < \ldots < i_m \leq n\}$ and $C\,'(n,m) = \{1 \leq i_1 \neq \ldots \neq i_m \leq n\}$ and $[n]^{-m} := \dfrac{(n-m)!}{n!}$. For further use throughout, we define

$$\tilde{h}_1(x) = \mathbb{E}(h(X_1, \ldots, X_m) - \theta | X_1 = x).$$

Definition. A sequence $X, X_1, X_2, \ldots,$ of i.i.d. random variables is said to be in the domain of attraction of the normal law ($X \in DAN$) if there exist sequences of constants A_n and $B_n > 0$ such that, as $n \to \infty$,

$$\frac{\sum_{i=1}^{n} X_i - A_n}{B_n} \longrightarrow_d N(0,1).$$

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Remark 1. Furtherer to this definition of DAN, it is known that A_n can be taken as $n\mathbb{E}(X)$ and $B_n = n^{1/2}\ell_X(n)$, where $\ell_X(n)$ is a slowly varying function at infinity (i.e., $\lim_{n\to\infty} \frac{\ell_X(nk)}{\ell_X(n)} = 1$ for any $k > 0$), defined by the distribution of X. Moreover, $\ell_X(n) = \sqrt{Var(X)} > 0$, if $Var(X) < \infty$, and $\ell_X(n) \to \infty$, as $n \to \infty$, if $Var(X) = \infty$. Also X has all moments less than 2, and the variance of X is positive, but need not be finite.

Noting that $\tilde{h}_1(X_1), \tilde{h}_1(X_2), \ldots$, are i.i.d. random variables with mean zero ($\mathbb{E}\tilde{h}_1(X_1) = 0$), Nasari (cf. [?]) observed that Proposition 2.1 of Csörgő, Szyszkowicz and Wang [CsSzW] (2004 [?]) (cf. also Theorem 1 of [CsSzW] 2003 [?]) reads as follows (cf. Lemma 2 in Nasari [?])

Lemma A . *As $n \to \infty$, the following statements are equivalent*:

(a) $\tilde{h}_1(X_1) \in DAN$;

There is a sequence of constants $B_n \nearrow \infty$, such that

(b) $\dfrac{\sum_{i=1}^{[nt_0]} \tilde{h}_1(X_i)}{B_n} \longrightarrow_d N(0, t_0)$ *for* $t_0 \in (0, 1]$;

(c) $\dfrac{\sum_{i=1}^{[nt]} \tilde{h}_1(X_i)}{B_n} \longrightarrow_d W(t)$ *on* $(D[0, 1], \rho)$, *where ρ is the sup-norm metric for functions in $D[0, 1]$ and $\{W(t), 0 \le t \le 1\}$ is a standard Wiener process*;

(d) *On an appropriate probability space for $X_1, X_2, \ldots$, we can construct a standard Wiener process $\{W(t), 0 \le t < \infty\}$ such that*

$$\sup_{0\le t\le 1} \left| \frac{\sum_{i=1}^{[nt]} \tilde{h}_1(X_i)}{B_n} - \frac{W(nt)}{n^{\frac{1}{2}}} \right| = o_P(1).$$

Here and throughout, B_n is as in Remark 1, from now on written as $B_n = n^{1/2}\ell(n)$, where $\ell(.)$, the slowly varying function at infinity, is defined by the distribution of the random variable $\tilde{h}_1(X_1)$ (cf. Remark 1).

Remark 2. The statement (c), whose notion will be used throughout, stands for the following functional central limit theorem (cf. Remark 2.1 in Csörgő, Szyszkowicz and Wang [CsSzw] (2004) [?]). On account of (d), as $n \to \infty$, we have

$$g(S_{[n.]}/V_n) \longrightarrow_d g(W(.))$$

for all $g : D = D[0, 1] \longrightarrow \mathbb{R}$ that are $(D, \mathfrak{D})$ measurable and ρ-continuous, or ρ-continuous except at points forming a set of Wiener measure zero on $(D, \mathfrak{D})$, where $\mathfrak{D}$ denotes the σ-field of subsets of D generated by the finite-dimensional subsets of D.

In view of (b) of Lemma A with $t_0 = 1$, Corollary 2.1 of [CsSzW] (2004 [?]), i.e., Raikov's theorem as stated and proved in Giné, Götze and Mason (1997

[?]), yields the following version of it in the present context.
Corollary A. *As $n \to \infty$, we have*

$$\frac{1}{n\ell^2(n)} \sum_{i=1}^{n} \tilde{h}_1^2(X_i) \longrightarrow_P 1.$$

Nasari [?] proved a projection approximation of $U_n$into sums of the i.i.d. random variables $\tilde{h}_1(X_1), \tilde{h}_1(X_2), \ldots$, that reads as follows (cf. Theorem 3 of [?]).

Theorem A. *If* $\mathbb{E}\ [|h(X_1,\ldots,X_m)|^{\frac{4}{3}} \log|h(X_1,\ldots,X_m)|\ ] < \infty$ *and* $\tilde{h}_1(X_1) \in DAN$, *then, as* $n \to \infty$, *we have*

$$\sup_{0\le t\le 1} \left| \frac{[nt]}{m} \frac{U_{[nt]}-\theta}{B_n} - \frac{\sum_{i=1}^{[nt]} \tilde{h}_1(X_i)}{B_n} \right| = o_P(1).$$

In view of Lemma A and Theorem A , Nasari [?] concluded his Theorem 2 that reads as follows.

Theorem B. *If*

(a) $\mathbb{E}\ [|h(X_1,\ldots,X_m)|^{\frac{4}{3}} \log|h(X_1,\ldots,X_m)|\ ] < \infty$ *and* $\tilde{h}_1(X_1) \in DAN$,

then, as $n \to \infty$, *we have*

(b) $\frac{[nt_0]}{m} \frac{U_{[nt_0]}-\theta}{B_n} \longrightarrow_d N(0,t_0)$, *where* $t_0 \in (0,1]$;

(c) $\frac{[nt]}{m} \frac{U_{[nt]}-\theta}{B_n} \longrightarrow_d W(t)$ *on* $(D[0,1],\rho)$, *where* ρ *is the sup-norm for functions in* $D[0,1]$ *and* $\{W(t), 0 \le t \le 1\}$ *is a standard Wiener process*;

(d) *On an appropriate probability space for* $X_1, X_2, \ldots$, *we can construct a standard Wiener process* $\{W(t), 0 \le t < \infty\}$ *such that*

$$\sup_{0\le t\le 1} \left| \frac{[nt]}{m} \frac{U_{[nt]}-\theta}{B_n} - \frac{W(nt)}{n^{\frac{1}{2}}} \right| = o_P(1).$$

We note in passing that the weak convergence result of part (c) of Theorem B for non degenerate U-statistics extend those obtained by Miller and Sen in 1972 (cf. Theorem 1 of [?])

Define the pseudo-selfnormalized U-process $U^*_{[nt]}$ as follows

$$U^*_{[nt]} = \begin{cases} 0 & , \quad 0 \le t < \frac{m}{n}, \\ \frac{U_{[nt]}-\theta}{V_n} & , \quad \frac{m}{n} \le t \le 1, \end{cases}$$

where [.] denotes the greatest integer function and $V_n^2 := \sum_{i=1}^n \tilde{h}_1^2(X_i)$. Combining Theorem A with Corollary A, Nasari (cf. [**?**]) inferred his Theorem 1 which reads as follows.

Theorem C. *If*

(a) $\mathbb{E}[\ |h(X_1,\ldots,X_m)|^{\frac{4}{3}} \log|h(X_1,\ldots,X_m)|\ ] < \infty$ *and* $\tilde{h}_1(X_1) \in DAN$,

then, as $n \to \infty$, *we have*

(b) $\dfrac{[nt_0]}{m}\ U^*_{[nt_0]} \to_d\ N(0,t_0),$ *for* $t_0 \in (0,1]$;

(c) $\dfrac{[nt]}{m}\ U^*_{[nt]}\ \to_d\ W(t)$ *on* $(D[0,1],\rho)$, *where* ρ *is the sup-norm for functions in* $D[0,1]$ *and* $\{W(t), 0 \le t \le 1\}$ *is a standard Wiener process*;

(d) *On an appropriate probability space for* $X_1, X_2, \ldots,$ *we can construct a standard Wiener process* $\{W(t), 0 \le t < \infty\}$ *such that*

$$\sup_{0 \le t \le 1} \left| \frac{[nt]}{m}\ U^*_{[nt]} - \frac{W(nt)}{n^{\frac{1}{2}}} \right| = o_P(1).$$

We note that in the light of Corollary A, a similarly pseudo-selfnormalized version of Lemma A is also immediate (cf. Lemma 1 in Nasari [**?**]). Moreover, these two lemmas, i.e., Lemmas 1 and 2 in Nasari [**?**], respectively coincide with Theorem 1 of [CsSzW] (2003 [**?**]), and with Proposition 2.1 of [CsSzW] (2004 [**?**]). Thus Theorems B and C with $m \ge 2$ amount to begin extensions of Theorem 1 of [CsSzW] (2003 [**?**]) to U-statistics of order $m \ge 2$.

While, in view of Raikov's theorem as in Corollary A, Theorems B and C are equivalent, Theorem C as stated constitutes a significant first step toward studentizing U-statistics for the sake of establishing asymptotic confidence intervals for θ in a nonparametric manner (cf. Theorem 1 and Corollary 1 of the next session that, in turn, leads to Main Theorem of this exposition). The pseudo-selfnormalizing sequence V_n of Theorem C still depends on the distribution function F that can not usually assumed to be known. Hence our Theorem 1 in this exposition.

2 Statement of the results

For $i = 1, \ldots, n$, let U^i_{n-1} be the *jackknifed* version of U_n based on $X_1, \ldots, X_{i-1}$, $X_{i+1}, \ldots, X_n$, defined as follows.

$$U^i_{n-1} = \frac{1}{n-1()m} \sum_{\substack{1 \le j_1 < \ldots < j_m \le n \\ j_1,\ldots,j_m \ne i}} h(X_{j_1}, \ldots, X_{j_m}).$$

Also define the *Studentized* U-process as follows.

$$U_{[nt]}^{stu} = \begin{cases} 0, & 0 \le t < \dfrac{m}{n}, \\ \dfrac{U_{[nt]} - \theta}{\sqrt{(n-1)\sum_{i=1}^{n}(U_{n-1}^{i} - U_n)^2}}, & \dfrac{m}{n} \le t \le 1. \end{cases}$$

Remark 3. Unlike the U-processes in Theorems B and C, apart from the parameter θ of interest, $U_{[nt]}^{stu}$ is completely computable, based on the observations $X_1, \dots, X_n$.

Under a slightly stronger moment condition, which is the price we pay for the normalization involved in $U_{[nt]}^{stu}$, the Studentized companion of Theorems B and C reads as follows.

Main Theorem. *If*

(a) $\mathbb{E}|h(X_1, \dots, X_m)|^{\frac{5}{3}} < \infty$ *and* $\tilde{h}_1(X_1) \in DAN$,

then, as $n \to \infty$, *we have*

(b) $[nt_0]\ U_{[nt_0]}^{stu} \to_d \ N(0, t_0),$ *for* $t_0 \in (0, 1]$;

(c) $[nt]\ U_{[nt]}^{stu} \to_d \ W(t)$ *on* $(D[0,1], \rho)$, *where* ρ *is the sup-norm for functions in* $D[0, 1]$ *and* $\{W(t), 0 \le t \le 1\}$ *is a standard Wiener process*;

(d) *On an appropriate probability space for* $X_1, X_2, \dots,$ *we can construct a standard Wiener process* $\{W(t), 0 \le t < \infty\}$ *such that*

$$\sup_{0 \le t \le 1} \left| [nt]\ U_{[nt]}^{stu} - \frac{W(nt)}{n^{\frac{1}{2}}} \right| = o_P(1).$$

In view of Theorems B and C and on account of Raikov's theorem (cf. Corollary A), which via (b) of Lemma A with $t_0 = 1$ in this context states that, as $n \to \infty$, $\dfrac{1}{n\ \ell^2(n)} \displaystyle\sum_{i=1}^{n} \tilde{h}_1^2(X_i) \to_P 1$, in order to prove Main Theorem it suffices to prove the following result.

Theorem 1. *If* $\mathbb{E}|h(X_1, \dots, X_m)|^{\frac{5}{3}} < \infty$ *and* $\tilde{h}_1(X_1) \in DAN$, *then, as* $n \to \infty$,

$$\left| \frac{(n-1)}{m^2\ \ell^2(n)} \sum_{i=1}^{n} (U_{n-1}^{i} - U_n)^2 - \frac{1}{n\ \ell^2(n)} \sum_{i=1}^{n} \tilde{h}_1^2(X_i) \right| = o_P(1).$$

Consequently, the latter approximation combined with Corollary A yields a Raikov type result for the distribution free jackkifed version of U-statistics which is of interest on its own (cf. Remark 4).

Corollary 1. *If* $\mathbb{E}|h(X_1,\ldots,X_m)|^{\frac{5}{3}} < \infty$ *and* $\tilde{h}_1(X_1) \in DAN$*, then, as* $n \to \infty$,

$$\frac{(n-1)}{m^2\ \ell^2(n)}\ \sum_{i=1}^{n}(U^i_{n-1} - U_n)^2 \longrightarrow_P 1.$$

Combining now Corollary 1 with Theorem B we arrive at Main Theorem of this paper.

Remark 4. When $\mathbb{E}\, h^2(X_1,\ldots,X_m) < \infty$, which in turn implies that $\mathbb{E}\tilde{h}^2_1(X_1) < \infty$, then $\ell^2(n) = \mathbb{E}\tilde{h}^2_1(X_1) > 0$ and, as $n \to \infty$, Corollary 1 implies that

$$\frac{(n-1)}{m^2}\ \sum_{i=1}^{n}(U^i_{n-1} - U_n)^2 \longrightarrow_P \mathbb{E}\tilde{h}^2_1(X_1).$$

The latter version of Corollary 1 coincides with one of the result obtained by Arvesen [**?**] who extended the idea of the so-called (by Tukey) pseudo- values to U-statistics and studied the asymptotic distribution of non-degenerate U-statistics via jackknifing.

Remark 5. When $m = 1$, the projection $\tilde{h}_1(X_1)$ will coincide with $h(X_1) - \theta$, then Main Theorem corresponds to Corollary 5 of [CsSzW] (2008 [**?**]) on taking the weight function $q = 1$ for the therein studied Studentized process $T_{n,t}(X-\mu)$, i.e., when $m = 1$, then the studentized U-process $U^{stu}_{[nt]}$ coincides with $T_{n,t}(X-\mu)$. Hence in this exposition we shall state our proofs for $m \geq 2$. Also when $m = 2$, the two conditions in (a) of Main Theorem as well as the idea of its proof by truncation, coincide with the corresponding ones of Theorem 2 of [CsSzW] (2008b [**?**]) on weighted approximations for Studentized U-type processes .

3 Proofs

To prove Theorem 1, it suffices to show that as $n \to \infty$,

$$\left|(n-1)\ \sum_{i=1}^{n}(U^i_{n-1} - U_n)^2 - \frac{m^2}{n}\sum_{i=1}^{n}\tilde{h}^2_1(X_i)\right| = o_P(1). \tag{1}$$

Before proving (1) we do some simplifications as follows.

$(n-1)\ \sum_{i=1}^{n}(U^i_{n-1} - U_n)^2$

$$= (n-1)\sum_{i=1}^{n}\left[\frac{\binom{n}{m}}{\binom{n-1}{m}}\ U_n - \ \frac{1}{\binom{n-1}{m}}\sum_{\substack{1\leq j_1<\ldots<j_{m-1}\leq n\\ j_1,\ldots,j_{m-1}\neq i}} h(X_i, X_{j_1}\ldots, X_{j_{m-1}}) - \ U_n\right]^2$$

$$= (n{-}1) \sum_{i=1}^{n} \left[\frac{1}{\binom{n-1}{m}} \sum_{\substack{1 \le j_1 < \ldots < j_{m-1} \le n \\ j_1, \ldots, j_{m-1} \ne i}} h(X_i, X_{j_1} \ldots, X_{j_{m-1}}) - \Big(\frac{\binom{n}{m}}{\binom{n-1}{m}} - 1 \Big) \, U_n \right]^2$$

$$= (n{-}1) \sum_{i_1=1}^{n} \left[\frac{m}{n-m} \Big(\frac{1}{\binom{n-1}{m-1}} \sum_{\substack{1 \le i_2 < \ldots < i_m \le n \\ i_2, \ldots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) - U_n \Big) \right]^2$$

$$= \frac{m^2(n-1)}{(n-m)^2} \sum_{i_1=1}^{n} \left[\frac{1}{\binom{n-1}{m-1}} \sum_{\substack{1 \le i_2 < \ldots < i_m \le n \\ i_2, \ldots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) - U_n \right]^2 \qquad (*)$$

$$= \frac{m^2(n-1)}{(n-m)^2} \sum_{i_1=1}^{n} \left[\frac{1}{\binom{n-1}{m-1}} \sum_{\substack{1 \le i_2 < \ldots < i_m \le n \\ i_2, \ldots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) \right]^2 + \frac{m^2 n(n-1)}{(n-m)^2} \, U_n^2$$

$$- 2 \, \frac{m^2(n-1)}{(n-m)^2} \, U_n \, \frac{1}{\binom{n-1}{m-1}} \sum_{i_1=1}^{n} \sum_{\substack{1 \le i_2 < \ldots < i_m \le n \\ i_2, \ldots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m})$$

$$= \frac{m^2(n-1)}{(n-m)^2} \sum_{i_1=1}^{n} \left[\frac{1}{\binom{n-1}{m-1}} \sum_{\substack{1 \le i_2 < \ldots < i_m \le n \\ i_2, \ldots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) \right]^2 + \frac{m^2 n(n-1)}{(n-m)^2} \, U_n^2$$

$$- 2 \, \frac{m^2(n-1)}{(n-m)^2} \, U_n \, \frac{1}{(m-1)! \, \binom{n-1}{m-1}} \sum_{i_1=1}^{n} \sum_{\substack{1 \le i_2 \ne \ldots \ne i_m \le n \\ i_2, \ldots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m})$$

$$= \frac{m^2(n-1)}{(n-m)^2} \sum_{i_1=1}^{n} \left[\frac{1}{\binom{n-1}{m-1}} \sum_{\substack{1 \le i_2 < \ldots < i_m \le n \\ i_2, \ldots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) \right]^2 + \frac{m^2 n(n-1)}{(n-m)^2} \, U_n^2$$

$$- 2 \, \frac{m^2(n-1)}{(n-m)^2} \, U_n \, \frac{1}{(m-1)! \, \binom{n-1}{m-1}} \sum_{1 \le i_1 \ne i_2 \ne \ldots \ne i_m \le n} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m})$$

$$= \frac{m^2(n-1)}{(n-m)^2} \sum_{i_1=1}^{n} \left[\frac{1}{\binom{n-1}{m-1}} \sum_{\substack{1 \le i_2 < \ldots < i_m \le n \\ i_2, \ldots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) \right]^2 + \frac{m^2 n(n-1)}{(n-m)^2} \, U_n^2$$

- 2 $\dfrac{m^2 n(n-1)}{(n-m)^2}\, U_n^2$

$$= \frac{m^2(n-1)}{(n-m)^2}\sum_{i_1=1}^{n}\left[\frac{1}{\binom{n-1}{m-1}}\sum_{\substack{1\le i_2<\ldots<i_m\le n\\ i_2,\ldots,i_m\neq i_1}} h(X_{i_1},X_{i_2},\ldots,X_{i_m})\right]^2 - \frac{m^2 n(n-1)}{(n-m)^2}\, U_n^2 \tag{2}$$

Remark 6. In view of (*) in what will follow without loss of generality we may and shall assume that $\theta = 0$.

In view of (2) to prove (1) it will be enough to prove the following two propositions.

Proposition 1. *If* $\mathbb{E}|h(X_1,\ldots,X_m)|^{\frac{5}{3}} < \infty$, *then, as* $n \to \infty$,

$$U_n^2 \longrightarrow 0 \quad a.s.$$

Proof of Proposition 1

The proof this theorem follows from the SLLN for U-statistics (cf. for example Serfling [**?**]).

Proposition 2. *If* $\mathbb{E}|h(X_1,\ldots,X_m)|^{\frac{5}{3}} < \infty$ and $\tilde{h}_1(X_1) \in DAN$ *then, as* $n \to \infty$,

$$\left|\frac{(n-1)}{(n-m)^2}\sum_{i_1=1}^{n}\left[\frac{1}{\binom{n-1}{m-1}}\sum_{\substack{1\le i_2<\ldots<i_m\le n\\ i_2,\ldots,i_m\neq i_1}} h(X_{i_1},X_{i_2},\ldots,X_{i_m})\right]^2 - \frac{1}{n}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)\right|$$

$$= o_P(1).$$

Proof of Proposition 2
In what will follow $a_n \approx b_n$ stands for the asymptotic equivalency of the numerical sequences $(a_n)_n$ and $(b_n)_n$, i.e., as $n \to \infty$, $\dfrac{a_n}{b_n} \to 1$.

To prove Proposition 2 observe that

$$\frac{(n-1)}{(n-m)^2}\sum_{i_1=1}^{n}\left[\frac{1}{\binom{n-1}{m-1}}\sum_{\substack{1\le i_2<\ldots<i_m\le n\\ i_2,\ldots,i_m\neq i_1}} h(X_{i_1},X_{i_2},\ldots,X_{i_m})\right]^2$$

$$= \frac{(n-1)}{(n-m)^2} \sum_{i_1=1}^{n} \left[[n-1]^{-m+1} \sum_{\substack{1 \le i_2 \ne \ldots \ne i_m \le n \\ i_2, \ldots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) \right]^2$$

$$\approx [n]^{-2m+1} \sum_{i_1=1}^{n} \left[\sum_{\substack{1 \le i_2 \ne \ldots \ne i_m \le n \\ i_2, \ldots, i_m \ne i_1}} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) \right]^2$$

$$= [n]^{-2m+1} \sum_{1 \le i_1 \ne \ldots \ne i_m \le n} h^2(X_{i_1}, \ldots, X_{i_m})$$

$$+ [n]^{-2m+1} \sum_{j=2}^{m-1} \sum_{1 \le i_1 \ne \ldots \ne i_{2m-j} \le n} h(X_{i_1}, \ldots, X_{i_j}, X_{i_{j+1}}, \ldots, X_{i_m}) \times h(X_{i_1}, \ldots, X_{i_j}, X_{i_{m+1}}, \ldots, X_{i_{2m-j}})$$

$$+ [n]^{-2m+1} \sum_{1 \le i_1 \ne \ldots \ne i_{2m-1} \le n} h(X_{i_1}, X_{i_2}, \ldots, X_{i_m}) \, h(X_{i_1}, X_{i_{m+1}}, \ldots, X_{i_{2m-1}}).$$

The first term and the second one, which obviously does not appear when $m = 2$, in the latter equality will be seen to be negligible in probability (cf. Propositions 3 and 4), thus the third term becomes the main term that will play the main role in establishing Proposition 2.

To complete the proof of Proposition 2 we shall state and prove the next three results, namely Propositions 3, 4 and Theorem 2.

Proposition 3. *If* $\mathbb{E}|h(X_1, \ldots, X_m)|^{\frac{5}{3}} < \infty$, *then, as* $n \to \infty$,

$$[n]^{-2m+1} \sum_{1 \le i_1 \ne \ldots \ne i_m \le n} h^2(X_{i_1}, \ldots, X_{i_m}) \to 0 \quad a.s.$$

Proof of Proposition 3

From the fact that for $m \ge 2$, $\dfrac{2m}{2m-1} < \dfrac{5}{3}$, it follows that

$$\mathbb{E} \left| h^2(X_1, \ldots, X_m) \right|^{\frac{m}{2m-1}} = \mathbb{E} \left| h(X_1, \ldots, X_m) \right|^{\frac{2m}{2m-1}} < \infty.$$

By this the proof of Proposition 3 follows from Theorem 1 of [**?**].

Proposition 4. For $m \ge 3$, *If* $\mathbb{E}|h(X_1, \ldots, X_m)|^{\frac{5}{3}} < \infty$, *then, as* $n \to \infty$,

$$[n]^{-2m+1} \sum_{j=2}^{m-1} \sum_{1 \le i_1 \ne \ldots \ne i_{2m-j} \le n} h(X_{i_1}, \ldots, X_{i_j}, X_{i_{j+1}}, \ldots, X_{i_m})$$

$$\times\, h(X_{i_1},\dots,X_{i_j},X_{i_{m+1}},\dots,X_{i_{2m-j}}) = o_P(1).$$

Proof of Proposition 4

In order to prove Proposition 4 it suffices to show that as $n\to\infty$, for $j = 2,\dots,m-1$, we have

$$[n]^{-2m+1} \sum_{1\le i_1\neq\ldots\neq i_{2m-j}\le n} h(X_{i_1},\dots,X_{i_j},X_{i_{j+1}},\dots,X_{i_m}) \times h(X_{i_1},\dots,X_{i_j},X_{i_{m+1}},\dots,X_{i_{2m-j}}) = o_P(1).$$

Since the proof of the latter relation can be done by modifying, mutatis mutandis (cf. Appendix), that of the next theorem, i.e., Theorem 2, hence the detailed proof is given in Appendix.

Theorem 2. *If* $\mathbb{E}|h(X_1,\dots,X_m)|^{\frac{5}{3}} < \infty$ and $\tilde{h}_1(X_1)\in DAN$ *then, as* $n\to\infty$,

$$\Big|\, [n]^{-2m+1} \sum_{1\le i_1\neq\ldots\neq i_{2m-1}\le n} h(X_{i_1},X_{i_2},\dots,X_{i_m}) h(X_{i_1},X_{i_{m+1}},\dots,X_{i_{2m-1}}) - \frac{1}{n}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)\,\Big| = o_P(1).$$

Proof of Theorem 2

Before stating the proof of Theorem 2 we need the following definition and lemma which will play a crucial role in our proofs.

Definition. The Borel- measurable function $L(x_1,\dots,x_m):\mathbb{R}^m\to\mathbb{R}$, $m\ge 2$, with mean $\mu=\mathbb{E}L(X_1,\dots,X_m)$, is said to be *degenerate* if for every proper subset $\{\alpha_1,\dots,\alpha_j\}$ of $\{1,\dots,m\}$, $j=1,\dots,m-1$, we have

$$\mathbb{E}\big(L(X_1,\dots,X_m)-\mu|X_{\alpha_1},\dots,X_{\alpha_j}\big) = 0 \;\; a.s.$$

We note in passing that if L were symmetric in its arguments, then the associated U-statistic with such a kernel would be a complete degenerate one. Hence our terminology for L in this definition.

Lemma 1. *If* $L:\mathbb{R}^m\to\mathbb{R}$, $m\ge 2$, *is degenerate with mean* $\mu=\mathbb{E}L(X_1,\dots,X_m)$ *and* $\mathbb{E}L^2(X_1,\dots,X_m)<\infty$, *then,*

$$\mathbb{E}\Big(\, [n]^{-m} \sum_{1\le i_1\neq\ldots\neq i_m\le n} (L(X_{i_1},\dots,X_{i_m})-\mu)\,\Big)^2 \le\; [n]^{-m}\,\mathbb{E}\,\big(L(X_1,\dots,X_m)-\mu\big)^2.$$

Proof of Lemma 1

Let $\hat{L}_{1\ldots m} := \dfrac{1}{m!}\sum_{C_m} L_{\sigma_1\ldots\sigma_m}$, where $L_{\sigma_1\ldots\sigma_m} := L(X_{\sigma_1},\dots,X_{\sigma_m})$ and C_m

denotes the set of all permutations $\sigma_1, \dots, \sigma_m$ of $1, \dots, m$. It is clear that

$$\sum_{1 \le i_1 \ne \ldots \ne i_m \le n} (\hat{L}_{i_1,\dots,i_m} - \mu) = \sum_{1 \le i_1 \ne \ldots \ne i_m \le n} (L_{i_1,\dots,i_m} - \mu).$$

Now write

$$\mathbb{E}\,\Big(\,[n]^{-m} \textstyle\sum_{1 \le i_1 \ne \ldots \ne i_m \le n} (\hat{L}_{i_1,\dots,i_m} - \mu)\,\Big)^2$$

$$= ([n]^{-m})^2 \textstyle\sum_{1 \le i_1 \ne \ldots \ne i_m \le n} \mathbb{E}(\hat{L}_{i_1,\dots,i_m} - \mu)^2$$

$$+ ([n]^{-m})^2 \textstyle\sum_{j=1}^{m-1} \sum_{1 \le i_1 \ne \ldots \ne i_{2m-j} \le n} \mathbb{E}[\,(\hat{L}_{i_1,\dots,i_j,i_{j+1},\dots,i_m} - \mu)$$

$$\times\,(\hat{L}_{i_1,\dots,i_j,i_{m+1},\dots,i_{2m-j}} - \mu)]$$

$$+ ([n]^{-m})^2 \textstyle\sum_{1 \le i_1 \ne \ldots \ne i_{2m} \le n} \mathbb{E}\,\big(\,(\hat{L}_{i_1,\dots,i_m} - \mu)\,(\hat{L}_{i_{m+1},\dots,i_{2m}} - \mu)\,\big)$$

$$= \;[n]^{-m}\;\mathbb{E}\;(\hat{L}_{1,\dots,m} - \mu)^2$$

$$+ ([n]^{-m})^2 \textstyle\sum_{j=1}^{m-1} \sum_{1 \le i_1 \ne \ldots \ne i_{2m-j} \le n} \mathbb{E}\{\mathbb{E}[\hat{L}_{i_1,\dots,i_j,i_{j+1},\dots,i_m} - \mu \mid X_{i_1}, \dots, X_{i_j}]$$

$$\times\;\mathbb{E}[\hat{L}_{i_1,\dots,i_j,i_{m+1},\dots,i_{2m-j}} - \mu \mid X_{i_1}, \dots, X_{i_j}]\}$$

$$+ ([n]^{-m})^2 \textstyle\sum_{1 \le i_1 \ne \ldots \ne i_{2m} \le n} \mathbb{E}(\hat{L}_{i_1,\dots,i_m} - \mu)\;\mathbb{E}(\hat{L}_{i_{m+1},\dots,i_{2m}} - \mu)$$

$$= \;[n]^{-m}\;\mathbb{E}\;(\hat{L}_{1,\dots,m} - \mu)^2$$

$$\le [n]^{-m}\;\mathbb{E}\;(L_{1,\dots,m} - \mu)^2.$$

The last inequality results from a well known inequality for sums of random variables followed by an application of Cauchy inequality provided that $\mathbb{E}L^2_{\sigma_1,\dots,\sigma_m} = \mathbb{E}L^2_{1\ldots m}$.

It is easy to observe that when L is symmetric in its arguments, the inequality in Lemma 1 becomes equality.

For further use in this proof, we consider the following setup:

$$h_{1\ldots m} := \;h(X_1, \dots, X_m),$$

$$h^{(m)}_{1\ldots m} := \;h_{1\ldots m}\;\mathbf{1}_{(|h| \le n^{\frac{3m}{5}})},$$

$$h^{*}_{12\ldots 2m-1} := \;h^{(m)}_{12\ldots m}\;h^{(m)}_{1m+1\ldots 2m-1},$$

$$\tilde{h}^{(m)}_1(x) := \;\mathbb{E}\;(\;h^{(m)}_{1\ldots m} | X_1 = x\;),$$

$$h^{(j)}_{1\ldots m} := \ h^{(m)}_{1\ldots m} \ \mathbf{1}_{(|h^{(m)}| \le n^{\frac{3j}{5}})}, \ j = 1, \ldots, m-1,$$
$$h^{(0)}_{1\ldots m} := \ h^{(m)}_{1\ldots m} \ \mathbf{1}_{(|h^{(m)}| \le \log(n))},$$
$$h^{(\ell)}_{1\ldots m} := \ h^{(m)}_{1\ldots m} \ \mathbf{1}_{(|\tilde{h}^{(m)}_1(x)| \le n^{1/2} \ \ell(n))},$$

where $\mathbf{1}_A$ denotes the indicator function of the set A and $\ell(.)$ is a slowly varying function at infinity associated to $\tilde{h}_1(X_1)$.

In view of the above set up, observe that as $n \to \infty$

$$\mathbb{P}\Big(\sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} h_{i_1 i_2 \ldots i_m} \, h_{i_1 i_{m+1} \ldots i_{2m-1}} \neq \sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} h^{(m)}_{i_1 i_2 \ldots i_m} \, h^{(m)}_{i_1 i_{m+1} \ldots i_{2m-1}} \Big)$$

$$\le \ n^m \ \mathbb{P} \ (\ |h_{1\ldots m}| > \ n^{\frac{3m}{5}} \ )$$
$$\le \mathbb{E} \ [\ |h_{1\ldots m}|^{\frac{5}{3}} \ 1_{(|h_{1\ldots m}| > n^{\frac{3m}{5}})} \ ] \longrightarrow 0.$$

Hence the asymptotic equivalency of the statistic of Theorem 2 and its truncated version i.e., $\sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} \ h^{(m)}_{i_1 i_2 \ldots i_m} \ h^{(m)}_{i_1 i_{m+1} \ldots i_{2m-1}}$ in probability.

Having the asymptotic equivalency of the original statistic and its truncated version, to prove Theorem 2, we shall proceed by working with the truncated version. Extending the idea of Hoffeding procedure to represent U-statistics in terms of complete *degenerate* ones (cf. for example [**?**]), in our context in which due to lack of symmetry, our statistic of interest i.e., $\sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} \ h^{(m)}_{i_1 i_2 \ldots i_m} \ h^{(m)}_{i_1 i_{m+1} \ldots i_{2m-1}}$ is not a U-statistic, by adding and subtracting required terms, we shall create a sequence of *degenerate* statistics. Then by employing proper new truncations and applying Lemma 1 we conclude the asymptotic negligibility of all of these degenerate statistics in probability (cf. Propositions 5, 6 and 7) except for the last group of them which are of the form of sums of i.i.d. random variables (cf. Remark 7). Among those the latter mentioned just one (cf. part (b) of Proposition 8) will asymptotically in probability coincide $\displaystyle\frac{1}{n}\sum_{i=1}^{n} \tilde{h}^2_1(X_i)$ and that will complete the proof of Theorem 2.

Now by adding and subtracting required terms we write

$$\sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} h^*_{i_1 \ldots i_{2m-1}}$$

$$= \sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} \Big\{ \sum_{d=1}^{2m-1} (-1)^{2m-1-d} \sum_{1 \le j_1 < \ldots < j_d \le 2m-1} \mathbb{E}(h^*_{i_1 \ldots i_{2m-1}} - \mathbb{E}(h^*_{i_1 \ldots i_{2m-1}}) | \ X_{i_{j_1}}, \ldots X_{i_{j_d}})$$

$$+ \sum_{c=1}^{2m-2} \ \sum_{1 \le k_1 < \ldots < k_c \le 2m-1} \ \sum_{d=1}^{c} (-1)^{c-d} \sum_{1 \le j_1 < \ldots < j_d \le c} \mathbb{E}(h^*_{i_1 \ldots i_{2m-1}} - \mathbb{E}(h^*_{i_1 \ldots i_{2m-1}}) | \ X_{i_{k_{j_1}}}, \ldots X_{i_{k_{j_d}}})$$

$$+ \ \mathbb{E} \ (h^*_{i_1 \ldots i_{2m-1}}) \Big\}$$

$$:= \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V(i_1,\ldots,i_{2m-1}) + \sum_{c=1}^{2m-2} \sum_{1\le k_1<\ldots<k_c\le 2m-1} \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V(i_{k_1},\ldots,i_{k_c})$$

$+ \ \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} \mathbb{E}(h^*_{i_1\ldots i_{2m-1}})$.

Proposition 5. *If* $\mathbb{E}\,|h_{1\ldots m}|^{\frac{5}{3}} < \infty$, *then, as* $n\to\infty$,

$$[n]^{-2m+1} \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V(i_1,\ldots,i_{2m-1}) \ = o_P(1).$$

Proof of Proposition 5

For throughout use K will be a positive constant that may be different at each stage.

Since $V(i_1,\ldots,i_{2m-1})$ posses the property of *degeneracy* we can apply Lemma 1 for the associated statistics and write, for $\epsilon>0$,

$$\begin{aligned}
&\mathbb{P}\ (\ |\ [n]^{-2m+1} \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V(i_1,\ldots,i_{2m-1})| > \epsilon) \\
&\le \epsilon^{-2}\ \mathbb{E}\ [\ [n]^{-2m+1} \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V(i_1,\ldots,i_{2m-1})\ ]^2 \\
&\le \epsilon^{-2}\ [n]^{-2m+1}\ \mathbb{E}\ [\ V(1,\ldots,2m-1)\ ]^2 \\
&\le\ K\ \epsilon^{-2}\ [n]^{-2m+1}\ n^{2m-1}\ n^{-2m+1}\ \mathbb{E}\ [\ h^{(m)}_{12\ldots m}\ h^{(m)}_{1m+1\ldots 2m-1}\ ]^2 \\
&\quad \le\ K\ \epsilon^{-2}\ [n]^{-2m+1}\ n^{2m-1}\ n^{-2m+1}\ n^{\frac{7m}{5}}\ \mathbb{E}\ |\ h_{12\ldots m}\ |^{\frac{5}{3}} \\
&\quad\quad \longrightarrow 0,\ \text{as } n\to\infty.
\end{aligned}$$

The estimation for $m\ge 3$ that occurs in our next proposition does not appear, and hence not needed, when $m=2$.

Proposition 6. *For* $m\ge 3$, *if* $\mathbb{E}\ |\ h_{12\ldots m}\ |^{\frac{5}{3}} < \infty$, *then as,* $n\to\infty$

$$[n]^{-2m+1} \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V(i_{k_1},\ldots,i_{k_c}) \ = o_P(1),$$

where $c=3,\ldots,2m-2$ *and* $1\le k_1<\ldots<k_c\le 2m-1$.

Proof of Proposition 6

Based on the way $i_{k_1},\ldots,i_{k_c}$ are distributed between $h^{(m)}_{i_1i_2\ldots i_m}$ and $h^{(m)}_{i_1i_{m+1}\ldots i_{2m-1}}$ in two different cases when $k_1=1$ and $k_1\ne 1$, the proof is stated as follows.

Case $k_1=1$

Let s and t be respectively the number of elements of the sets $\{i_{k_1},\ldots,i_{k_c}\}\cap$

$\{i_1, i_2, \ldots, i_m\}$ and $\{i_{k_1}, \ldots, i_{k_c}\} \cap \{i_1, i_{m+1}, \ldots, i_{2m-1}\}$. It is clear that in this case, i.e., $k_1 = 1$, we have that $s, t \geq 1$ and $s + t = c + 1$. Now define

$$V^T(i_{k_1}, \ldots, i_{k_c}) = \sum_{d=1}^{c} (-1)^{c-d} \sum_{1 \leq j_1 < \ldots < j_d \leq c} \mathbb{E}(h^{*T}_{i_1 \ldots i_{2m-1}} - \mathbb{E}(h^{*T}_{i_1 \ldots i_{2m-1}}) \mid X_{i_{k_{j_1}}}, \ldots, X_{i_{k_{j_d}}}), \quad (3)$$

$$V^{T'}(i_{k_1}, \ldots, i_{k_c}) = \sum_{d=1}^{c} (-1)^{c-d} \sum_{1 \leq j_1 < \ldots < j_d \leq c} \mathbb{E}(h^{*T'}_{i_1 \ldots i_{2m-1}} - \mathbb{E}(h^{*T'}_{i_1 \ldots i_{2m-1}}) \mid X_{i_{k_{j_1}}}, \ldots, X_{i_{k_{j_d}}}), \quad (4)$$

where $h^{*T}_{i_1 \ldots i_{2m-1}} = h^{(s)}_{i_1 i_2 \ldots i_m} \, h^{(m)}_{i_1 i_{m+1} \ldots i_{2m-1}}$ and $h^{*T'}_{i_1 \ldots i_{2m-1}} = h^{(s)}_{i_1 i_2 \ldots i_m} \, h^{(t)}_{i_1 i_{m+1} \ldots i_{2m-1}}$.

Now observe that as $n \to \infty$

$$\mathbb{P}\,(\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V(i_{k_1}, \ldots, i_{k_c}) \neq \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V^{T'}(i_{k_1}, \ldots, i_{k_c})\,)$$

$$\leq \mathbb{P}\,(\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V(i_{k_1}, \ldots, i_{k_c}) \neq \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V^{T}(i_{k_1}, \ldots, i_{k_c})\,)$$
$$+ \mathbb{P}\,(\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V^T(i_{k_1}, \ldots, i_{k_c}) \neq \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V^{T'}(i_{k_1}, \ldots, i_{k_c})\,)$$

$$\leq \; n^s \, \mathbb{P}\,(\, |h^{(m)}_{12 \ldots m}| > \; n^{\frac{3s}{5}} \,) + \; n^t \, \mathbb{P}\,(\, |h^{(m)}_{1m+1 \ldots 2m-1}| > \; n^{\frac{3t}{5}} \,)$$

$$\leq \; \mathbb{E}\,[\, |h_{12 \ldots m}|^{\frac{5}{3}} \, \mathbf{1}_{(|h| > n^{\frac{3s}{5}})} \,] + \; \mathbb{E}\,[\, |h_{1m+1 \ldots 2m-1}|^{\frac{5}{3}} \, \mathbf{1}_{(|h| > n^{\frac{3t}{5}})} \,] \longrightarrow 0.$$

The latter relation suggests that $\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V(i_{k_1}, \ldots, i_{k_c})$ and $\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V^{T'}(i_{k_1}, \ldots, i_{k_c})$ are asymptotically equivalent in probability.

Since $V^{T'}(i_{k_1}, \ldots, i_{k_c})$ is *degenerate*, Markov inequality followed by an application of Lemma 1 yields,

$$\mathbb{P}\,(\, | \, [n]^{-2m+1} \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V^{T'}(i_{k_1}, \ldots, i_{k_c}) \, | \; > \epsilon \,)$$
$$\leq \; \epsilon^{-2} \, \mathbb{E}\,[\, [n]^{-2m+1} \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V^{T'}(i_{k_1}, \ldots, i_{k_c}) \,]^2$$
$$\leq \; K \epsilon^{-2} \, [n - (2m-1-c)]^{-c} \, \mathbb{E}\,[\, h^{(s)}_{12 \ldots m} \, h^{(t)}_{1m+1 \ldots 2m-1} \,]^2$$

$$\leq \; K \epsilon^{-2} \, [n - (2m-1-c)]^{-c} \, n^c \, n^{-c} \, n^{\frac{7(t+s)}{10}} \, \mathbb{E} \mid h_{12 \ldots m} \mid^{\frac{5}{3}}$$
$$\longrightarrow 0, \text{ as } n \to \infty.$$

The latter relation is true since when $c \geq 3$, we have $-c + \dfrac{7(t+s)}{10} < 0$.

Case $k_1 \neq 1$

Similarly to the previous case let s and t be respectively the number of elements of the sets $\{i_{k_1},\ldots,i_{k_c}\}\cap\{i_1,i_2,\ldots,i_m\}$ and $\{i_{k_1},\ldots,i_{k_c}\}\cap\{i_1,i_{m+1},\ldots,i_{2m-1}\}$. Clearly here we have $s,t\geq 0$ and $s+t=c$. It is obvious that in this case s,t can be zero but not simultaneously. More specifically, $(s=c,t=0)$ and $(s=0,t=c)$ can happen and due to their similarity we shall only treat $(s=c,t=0)$.

Let $V^{T}(i_{k_1},\ldots,i_{k_c})$ and $V^{T'}(i_{k_1},\ldots,i_{k_c})$ be of the forms respectively (3) and (4), where $h^{*T}_{i_1\ldots i_{2m-1}} = h^{(s)}_{i_1 i_2\ldots i_m}\ h^{(m)}_{i_1 i_{m+1}\ldots i_{2m-1}}$ and $h^{*T'}_{i_1\ldots i_{2m-1}} = h^{(s)}_{i_1 i_2\ldots i_m}\ h^{(t)}_{i_1 i_{m+1}\ldots i_{2m-1}}$.

Observe that as $n\to\infty$

$$\mathbb{P}\ \Big(\sum_{1\leq i_1\neq\ldots\neq i_{2m-1}\leq n} V(i_{k_1},\ldots,i_{k_c})\ \neq \sum_{1\leq i_1\neq\ldots\neq i_{2m-1}\leq n} V^{T'}(i_{k_1},\ldots,i_{k_c})\ \Big)$$

$$\leq \begin{cases} n^s\ \mathbb{P}\ (\ |h^{(m)}_{12\ldots m}|>\ n^{\frac{3s}{5}}\ )+\ n^t\ \mathbb{P}\ (\ |h^{(m)}_{1m+1\ldots 2m-1}|>\ n^{\frac{3t}{5}}\ ), & \text{s,t}>0,\ \text{s+t=c};\\ n^c\ \mathbb{P}(\ |h^{(m)}_{12\ldots m}|>\ n^{\frac{3c}{5}}\ )+\ \mathbb{P}\ (\ |h^{(m)}_{1m+1\ldots 2m-1}|>\ \log(n)\ ), & \text{s=c,t=0}\end{cases}$$

$$\leq \begin{cases} \mathbb{E}\ [\ |h_{12\ldots m}|^{\frac{5}{3}}\ \mathbf{1}_{(|h|>n^{\frac{3s}{5}})}\ ]+\ \mathbb{E}\ [\ |h_{1m+1\ldots 2m-1}|^{\frac{5}{3}}\ \mathbf{1}_{(|h|>n^{\frac{3t}{5}})}\ ], & \text{s,t}>0,\ \text{s+t=c};\\ \mathbb{E}\ [\ |h_{12\ldots m}|^{\frac{5}{3}}\ \mathbf{1}_{(|h|>n^{\frac{3c}{5}})}\ ]+\ \mathbb{P}\ (\ |h^{(m)}_{1m+1\ldots 2m-1}|>\ \log(n)\ ), & \text{s=c,t=0}\end{cases}$$

$\longrightarrow 0.$

Applying Markov inequality followed by an application Lemma 1 once again yields,

$$\mathbb{P}\ (\ |\ [n]^{-2m+1} \sum_{1\leq i_1\neq\ldots\neq i_{2m-1}\leq n} V^{T'}(i_{k_1},\ldots,i_{k_c})|>\epsilon)$$

$$\leq K\epsilon^{-2}\ [n-(2m-1-c)]^{-c}\ n^c\ n^{-c}\ \mathbb{E}\ [\ h^{(s)}_{12\ldots m}\ h^{(t)}_{1m+1\ldots 2m-1}\ ]^2$$

$$\leq \begin{cases} K\epsilon^{-2}\ [n-(2m-1-c)]^{-c}\ n^c\ n^{-c}\ n^{\frac{7c}{10}}\ \mathbb{E}|h_{12\ldots m}|^{\frac{5}{3}}, & \text{s,t}>0,\ \text{s+t=c};\\ K\epsilon^{-2}\ [n-(2m-1-c)]^{-c}\ n^c\ n^{-c}\ n^{\frac{7c}{10}}\ \log^{\frac{7}{6}}(n)\ \mathbb{E}|h_{12\ldots m}|^{\frac{5}{3}}, & \text{s=c,t=0}\end{cases}$$

$\longrightarrow 0$, as $n\to\infty$.

This completes the proof of Proposition 6.

Proposition 7. *If* $\mathbb{E}\ |h_{12\ldots m}|^{\frac{5}{3}}<\infty$ *and* $\tilde{h}_1(X_1)\in DAN$*, then, as* $n\to\infty$

$$[n]^{-2m+1} \sum_{1\leq i_1\neq\ldots\neq i_{2m-1}\leq n} V(i_{k_1},i_{k_2})=o_P(1),$$

where, $1 \le k_1 < k_2 \le 2m-1$.

Proof of Proposition 7

As it was the case in the proof of the last proposition, we shall state the proof for two cases $k_1 = 1$ and $k_1 \neq 1$ separately.

Case $k_1 = 1$

Again let s and t be respectively the number of elements of the sets $\{i_{k_1}, i_{k_2}\} \cap \{i_1, i_2, \ldots, i_m\}$ and $\{i_{k_1}, i_{k_2}\} \cap \{i_1, i_{m+1}, \ldots, i_{2m-1}\}$. It is clear that in this case we either have $(s = 2, t = 1)$ or $(s = 1, t = 2)$ which due to their similarity only $(s = 2, t = 1)$ will be treated as follows.

Define

$$V^T(i_{k_1}, i_{k_2}) = \sum_{d=1}^{2} (-1)^{2-d} \sum_{1 \le j_1 < \ldots < j_d \le 2} \mathbb{E}(h^{*T}_{i_1 \ldots i_{2m-1}} - \mathbb{E}(h^{*T}_{i_1 \ldots i_{2m-1}}) \mid x_{i_{k_{j_1}}}, \ldots, x_{i_{k_{j_d}}}),$$

$$V^{T'}(i_{k_1}, i_{k_2}) = \sum_{d=1}^{2} (-1)^{2-d} \sum_{1 \le j_1 < \ldots < j_d \le 2} \mathbb{E}(h^{*T'}_{i_1 \ldots i_{2m-1}} - \mathbb{E}(h^{*T'}_{i_1 \ldots i_{2m-1}}) \mid x_{i_{k_{j_1}}}, \ldots, x_{i_{k_{j_d}}}),$$

where $h^{*T}_{i_1 \ldots i_{2m-1}} = h^{(2)}_{i_1 i_2 \ldots i_m} \, h^{(m)}_{i_1 i_{m+1} \ldots i_{2m-1}}$ and $h^{*T'}_{i_1 \ldots i_{2m-1}} = h^{(2)}_{i_1 i_2 \ldots i_m} \, h^{(\ell)}_{i_1 i_{m+1} \ldots i_{2m-1}}$.

As $n \to \infty$, we have

$$\mathbb{P} \Big(\sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} V(i_{k_1}, i_{k_2}) \neq \sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} V^{T'}(i_{k_1}, i_{k_2}) \Big)$$

$$\le \mathbb{P} \Big(\sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} V(i_{k_1}, i_{k_2}) \neq \sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} V^{T}(i_{k_1}, i_{k_2}) \Big)$$

$$+ \mathbb{P} \Big(\sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} V^T(i_{k_1}, i_{k_2}) \neq \sum_{1 \le i_1 \neq \ldots \neq i_{2m-1} \le n} V^{T'}(i_{k_1}, i_{k_2}) \Big)$$

$$\le n^2 \, \mathbb{P}(\, |h^{(m)}_{12\ldots m}| > n^{6/5} \,) + n \, \mathbb{P}(\, |\tilde{h}^{(m)}_1(X_1)| > n^{1/2} \, \ell(n) \,)$$

$$\le \mathbb{E} \, [\, |h_{12\ldots m}|^{\frac{5}{3}} \, \mathbf{1}_{(|h| > n^{6/5})} \,] + n \, \mathbb{P}(\, |\tilde{h}^{(m)}_1(X_1)| > n^{1/2} \, \ell(n) \,)$$

$$:= I_1(n) + I_2(n).$$

It can be easily seen that as n tends to infinity $I_1(n) \to 0$.

To deal with $I_2(n)$ we write

$$n \, \mathbb{P} \, (\, |\tilde{h}^{(m)}_1(X_1)| > n^{1/2} \, \ell(n) \,)$$

$$\leq \; n \; \mathbb{P} \; (\; |\tilde{h}_1(X_1)| > \frac{n^{1/2} \; \ell(n)}{2} \;)$$

$$+ \; n \; \mathbb{P} \; (\; | \; \mathbb{E}(h_{1m+1 \ldots 2m-1} \; \mathbf{1}_{(|h|>n^{\frac{3m}{5}})} \; | \; X_1) \; | > \frac{n^{1/2} \; \ell(n)}{2})$$

$$\leq \; n \; \mathbb{P} \; (\; |\tilde{h}_1(X_1)| > \frac{n^{1/2} \; \ell(n)}{2} \;)$$

$$+ \; 2 \; n^{1/2} \; \ell^{-1}(n) \; \mathbb{E} \; [\; | \; h_{1m+1 \ldots 2m-1} \; | \; \mathbf{1}_{(|h|>n^{\frac{3m}{5}})} \;]$$

$$\leq \; n \; \mathbb{P} \; (\; |\tilde{h}_1(X_1)| > \frac{n^{1/2} \; \ell(n)}{2} \;)$$

$$+ \; 2 \; n^{1/2} \; n^{-\frac{2m}{5}} \; \ell^{-1}(n) \; \mathbb{E}| \; h_{1m+1 \ldots 2m-1} \; |^{\frac{5}{3}}$$

$$\longrightarrow 0, \; \text{as } n \to \infty.$$

The latter relation is true since $\tilde{h}_1(X_1) \in DAN$ and $m \geq 2$, and it means that $I_2(n) = o(1)$. Hence the asymptotic equivalency of $\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V(i_{k_1}, i_{k_2})$ and $\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V^{T'}(i_{k_1}, i_{k_2})$ in probability.

Before applying Lemma 1 for $[n]^{-2m+1} \; \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V^{T'}(i_{k_1}, i_{k_2})$, since we know that $k_1 = 1$ and $s = 2$, due to symmetry of $h_{i_1 i_2 \ldots i_m}$, without loss of generality we assume that $k_2 = 2$.

Now for $\epsilon > 0$, Markov inequality and Lemma 1 lead to

$$\mathbb{P}(\; | \; [n]^{-2m+1} \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-1} \leq n} V^{T'}(i_1, i_2) \; | > \epsilon \;)$$

$$\leq \; K \; \epsilon^{-2} \; [n-(2m-3)]^{-2} \; n^2 \; n^{-2} \; \mathbb{E}[\mathbb{E}(h^{(2)}_{12 \ldots m} h^{(\ell)}_{1m+1 \ldots 2m-1} - \mathbb{E}(h^{(2)}_{12 \ldots m} h^{(\ell)}_{1m+1 \ldots 2m-1}) | X_1, X_2)]^2$$

$$+ K \; \epsilon^{-2} \; [n-(2m-3)]^{-2} \; n^2 \; n^{-2} \; \mathbb{E}[\mathbb{E}(h^{(2)}_{12 \ldots m} h^{(\ell)}_{1m+1 \ldots 2m-1} - \mathbb{E}(h^{(2)}_{12 \ldots m} h^{(\ell)}_{1m+1 \ldots 2m-1}) | X_1)]^2$$

$$+ K \; \epsilon^{-2} \; [n-(2m-3)]^{-2} \; n^2 \; n^{-2} \; \mathbb{E}[\mathbb{E}(h^{(2)}_{12 \ldots m} h^{(\ell)}_{1m+1 \ldots 2m-1} - \mathbb{E}(h^{(2)}_{12 \ldots m} h^{(\ell)}_{1m+1 \ldots 2m-1}) | X_2)]^2$$

$$:= \; K \; \epsilon^{-2} \; [n - (2m-3)]^{-2} \; n^2 \; J_1(n)$$

$$+ \; K \; \epsilon^{-2} \; [n - (2m-3)]^{-2} \; n^2 \; J_2(n)$$

$$+ \; K \; \epsilon^{-2} \; [n - (2m-3)]^{-2} \; n^2 \; J_3(n).$$

Considering that as $n \to \infty$, $[n - (2m-3)]^{-2} \; n^2 \to 1$, we will show that $J_1(n), J_2(n), J_3(n) = o(1)$.

To deal with $J_1(n)$ write

$$J_1(n) \leq \; n^{-2} \; \mathbb{E}[\; \mathbb{E}(\; h^{(2)}_{12 \ldots m} h^{(\ell)}_{1m+1 \ldots 2m-1} \; | \; X_1, X_2) \;]^2$$

$$= \; n^{-2} \; \mathbb{E}[\; \mathbb{E}^2(h^{(2)}_{12 \ldots m} | \; X_1, X_2) \; \mathbb{E}^2(h^{(\ell)}_{1m+1 \ldots 2m-1} | \; X_1) \;]$$

$$= \; n^{-2} \; \mathbb{E}[\; \mathbb{E}^2(h^{(2)}_{12 \ldots m} | \; X_1, X_2) \; \mathbb{E}^2(h^{(m)}_{1m+1 \ldots 2m-1} | \; X_1) \; \mathbf{1}_{(|\tilde{h}^{(m)}_1(X_1)| \leq n^{1/2} \; \ell(n))} \;]$$

$$\leq \; n^{-1} \; \ell^2(n) \; \mathbb{E}[\; h^{(2)}_{12 \ldots m}]^2$$

$$\leq\ n^{-\frac{3}{5}}\ \ell^2(n)\ \mathbb{E}|\ h_{12\ldots m}|^{\frac{5}{3}}$$

$$\longrightarrow 0,\ \text{as } n\to\infty,$$

i.e., $J_1(n) = o(1)$. A similar argument yields, $J_2(n) = o(1)$, hence the details are omitted.

As for $J_3(n)$ we write

$$\begin{aligned} J_3(n) \leq\ & n^{-2}\,\mathbb{E}[\,\mathbb{E}(h^{(2)}_{12\ldots m}h^{(\ell)}_{1m+1\ldots 2m-1}\mid X_2)\,]^2 \\ =\ & n^{-2}\ \mathbb{E}\{\ \mathbb{E}[\mathbb{E}(h^{(2)}_{12\ldots m}h^{(\ell)}_{1m+1\ldots 2m-1}|X_1,\ldots,X_m)\mid X_2\ ]\}^2 \\ =\ & n^{-2}\ \mathbb{E}\{\ \mathbb{E}(h^{(2)}_{12\ldots m}|\ X_2)\ \mathbb{E}(h^{(\ell)}_{1m+1\ldots 2m-1}|\ X_1)\}^2 \\ \leq\ & n^{-\frac{3}{5}}\ \ell^2(n)\ \mathbb{E}|\ h_{12\ldots m}|^{\frac{5}{3}} \\ & \longrightarrow 0,\ \text{as } n\to\infty. \end{aligned}$$

The latter relation means that $J_3(n) = o(1)$. By this the proof of Proposition 7 when $k_1 = 1$ is complete.

At this stage we give the proof of Proposition 7 when $k_1 \neq 1$.

Case $k_1 \neq 1$

Once again let s and t be respectively the number of elements of the sets $\{i_{k_1}, i_{k_2}\} \cap \{i_1, i_2, \ldots, i_m\}$ and $\{i_{k_1}, i_{k_2}\} \cap \{i_1, i_{m+1}, \ldots, i_{2m-1}\}$. It is obvious that in this case the possibilities are either $s = t = 1$ or when $m \geq 3$, $(s = 2, t = 0)$ or $(s = 0, t = 2)$. We shall treat the cases $s = t = 1$ and $(s = 2, t = 0)$ when $m \geq 3$, separately as follows.

Case $k_1 \neq 1$*:* $s = t = 1$

We note that here we have $k_1 \in \{2, \ldots, m\}$ and $k_2 \in \{m+1, \ldots, 2m-1\}$.

Now define

$$V^T(i_{k_1}, i_{k_2}) = \sum_{d=1}^{2}(-1)^{2-d}\sum_{1\leq j_1<\ldots<j_d\leq 2}\mathbb{E}(h^{*^T}_{i_1\ldots i_{2m-1}} - \mathbb{E}(h^{*^T}_{i_1\ldots i_{2m-1}})\mid X_{i_{k_{j_1}}},\ldots,X_{i_{k_{j_d}}}),$$

$$V^{T'}(i_{k_1}, i_{k_2}) = \sum_{d=1}^{2}(-1)^{2-d}\sum_{1\leq j_1<\ldots<j_d\leq 2}\mathbb{E}(h^{*^{T'}}_{i_1\ldots i_{2m-1}} - \mathbb{E}(h^{*^{T'}}_{i_1\ldots i_{2m-1}})\mid X_{i_{k_{j_1}}},\ldots,X_{i_{k_{j_d}}}),$$

where $h^{*^T}_{i_1\ldots i_{2m-1}} = h^{(1)}_{i_1 i_2\ldots i_m}\ h^{(m)}_{i_1 i_{m+1}\ldots i_{2m-1}}$ and $h^{*^{T'}}_{i_1\ldots i_{2m-1}} = h^{(1)}_{i_1 i_2\ldots i_m}\ h^{(1)}_{i_1 i_{m+1}\ldots i_{2m-1}}$. Now observe that as $n\to\infty$ we have

$$\mathbb{P}\ (\sum_{1\leq i_1\neq\ldots\neq i_{2m-1}\leq n} V(i_{k_1}, i_{k_2})\ \neq \sum_{1\leq i_1\neq\ldots\neq i_{2m-1}\leq n} V^{T'}(i_{k_1}, i_{k_2})\ )$$

$$\leq \mathbb{P}\ (\sum_{1\leq i_1\neq\ldots\neq i_{2m-1}\leq n} V(i_{k_1}, i_{k_2})\ \neq \sum_{1\leq i_1\neq\ldots\neq i_{2m-1}\leq n} V^{T}(i_{k_1}, i_{k_2})\ )$$

$$+ \mathbb{P}\,(\sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V^T(i_{k_1},i_{k_2}) \;\ne \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V^{T'}(i_{k_1},i_{k_2})\,)$$

$$\le\; 2\;n\;\mathbb{P}\;(\;|h^{(m)}_{12\ldots m}| >\; n^{3/5}\;)$$
$$\le\; 2\;\mathbb{E}[\;|h_{12\ldots m}|^{\frac{5}{3}}\;\mathbf{1}_{(|h|>n^{3/5})}\;]$$
$$\longrightarrow 0.$$

In view of the latter relation we apply Lemma 1 to $[n]^{-2m+1}\sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V^{T'}(i_{k_1},i_{k_2})$ and we get

$$\mathbb{P}(\;|[n]^{-2m+1}\sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V^{T'}(i_{k_1},i_{k_2})| > \epsilon)$$
$$\le\; K\;\epsilon^{-2}\;[n-(2m-3)]^{-2}\;n^2\;n^{-2}\;\mathbb{E}(h^{(1)}_{12\ldots m}h^{(1)}_{1m+1\ldots 2m-1})^2$$
$$\le\; K\;\epsilon^{-2}\;[n-(2m-3)]^{-2}\;n^2\;n^{-2}\;n^{7/5}\mathbb{E}|h_{12\ldots m}|^{\frac{5}{3}}$$
$$\longrightarrow 0,\;\text{as } n\to\infty.$$

This completes the proof of Proposition 7 for the Case $k_1 \ne 1$ when $s = t = 1$.

Case $k_1 \ne 1$*: (* $m \ge 3$ *)* $s = 2, t = 0$

In this case we first note that $k_1, k_2 \in \{2,\ldots,m\}$. Now define

$$V^T(i_{k_1},i_{k_2}) = \sum_{d=1}^{2}(-1)^{2-d}\sum_{1\le j_1<\ldots<j_d\le 2}\mathbb{E}(h^{*T}_{i_1\ldots i_{2m-1}} - \mathbb{E}(h^{*T}_{i_1\ldots i_{2m-1}}) \mid x_{i_{k_{j_1}}},\ldots,x_{i_{k_{j_d}}}),$$

$$V^{T'}(i_{k_1},i_{k_2}) = \sum_{d=1}^{2}(-1)^{2-d}\sum_{1\le j_1<\ldots<j_d\le 2}\mathbb{E}(h^{*T'}_{i_1\ldots i_{2m-1}} - \mathbb{E}(h^{*T'}_{i_1\ldots i_{2m-1}}) \mid x_{i_{k_{j_1}}},\ldots,x_{i_{k_{j_d}}}),$$

where $h^{*T}_{i_1\ldots i_{2m-1}} = h^{(2)}_{i_1i_2\ldots i_m}\,h^{(m)}_{i_1i_{m+1}\ldots i_{2m-1}}$ and $h^{*T'}_{i_1\ldots i_{2m-1}} = h^{(2)}_{i_1i_2\ldots i_m}\,h^{(0)}_{i_1i_{m+1}\ldots i_{2m-1}}$.
Now observe that as $n\to\infty$

$$\mathbb{P}\,(\sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V(i_{k_1},i_{k_2}) \;\ne \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V^{T'}(i_{k_1},i_{k_2})\,)$$

$$\le \mathbb{P}\,(\sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V(i_{k_1},i_{k_2}) \;\ne \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V^{T}(i_{k_1},i_{k_2}\;)\,)$$
$$+ \mathbb{P}\,(\sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V^T(i_{k_1},i_{k_2}) \;\ne \sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V^{T'}(i_{k_1},i_{k_2})\,)$$
$$\le\; n^2\;\mathbb{P}(\;|h^{(m)}_{12\ldots m}| > n^{6/5}\;) + \;\mathbb{P}(\;|h^{(m)}_{1m+1\ldots 2m-1}| > \log(n)\;)$$
$$\le\; \mathbb{E}[\;|\;h_{12\ldots m}\;|^{\frac{5}{3}}\;\mathbf{1}_{(|h|>n^{6/5})}\;] + \;\mathbb{P}(\;|h_{1m+1\ldots 2m-1}| > \log(n)\;)$$

$\longrightarrow 0.$

The latter relation together with *degeneracy* of $V^{T'}(i_{k_1}, i_{k_2})$ enable us to use Lemma 1 once again and arrive at

$$\mathbb{P}(|\ [n]^{-2m+1} \sum_{1\le i_1\ne \ldots \ne i_{2m-1}\le n} V^{T'}(i_{k_1}, i_{k_2})\ | > \epsilon)$$

$$\le\ K\ \epsilon^{-2}\ [n-(2m-3)]^{-2}\ n^2\ n^{-2}\ \mathbb{E}(h^{(2)}_{12\ldots m}\ h^{(0)}_{1m+1\ldots 2m-1})^2$$

$$\le\ K\ \epsilon^{-2}\ [n-(2m-3)]^{-2}\ n^2\ n^{-\frac{3}{5}}\ \log^{7/6}(n)\ \mathbb{E}|h_{12\ldots m}|^{\frac{5}{3}}$$

$$\longrightarrow 0,\ \text{as}\ n\to\infty.$$

Now the proof of Proposition 7 is complete.

Remark 7. Before stating our next result we note in passing that when $k_1 = 1$ then $[n]^{-2m+1}\sum_{1\le i_1\ne\ldots\ne i_{2m-1}\le n} V(i_{k_1})$ is of the form

$$[n-(2m-2)]^{-1} \sum_{i_1\in\{1,\ldots,n\}/\{2,\ldots,2m-1\}}^{n} \mathbb{E}(h^{*}_{i_1 2\ldots 2m-1} - \mathbb{E}(h^{*}_{i_1 2\ldots 2m-1})\ |X_{i_1}),$$

otherwise, i.e., when for example $k_1 = 2$ it has the following form

$$[n-(2m-2)]^{-1} \sum_{i_2\in\{1,\ldots,n\}/\{1,3,\ldots,2m-1\}}^{n} \mathbb{E}(h^{*}_{1 i_2 3\ldots 2m-1} - \mathbb{E}(h^{*}_{1 i_2 3\ldots 2m-1})\ |X_{i_2}),$$

and so on for $k_1 \in \{2,\ldots,2m-1\}$.

Proposition 8. *If* $\mathbb{E}|h_{1\ldots m}|^{\frac{5}{3}} < \infty$ *and* $\tilde{h}_1(X_1) \in DAN$*, then, as* $n\to\infty$

(a) $[n]^{-2m+1}\sum_{1\le\ne i_1\ne\ldots\ne i_{2m-1}\le n} V(i_{k_1}) = o_P(1),$ *for* $k_1 \in \{2,\ldots,2m-1\}$,

(*b*)
$$|\ [n-(2m-2)]^{-1} \sum_{i\in\{1,\ldots,n\}/\{2,\ldots,2m-1\}}^{n} \mathbb{E}(h^{*}_{i2\ldots 2m-1} - \mathbb{E}(h^{*}_{i2\ldots 2m-1})\ |X_i)$$
$$+\,\mathbb{E}(h^{*}_{12\ldots 2m-1}) - \frac{1}{n}\sum_{i=1}^{n}\tilde{h}_1^2(X_i)\ | = o_P(1).$$

Proof of Proposition 8

First we give the proof of part (a). Due to similarities, we state the proof only for $k_1 = 2$.

Define

$$V^{T}(i_2) =\ \mathbb{E}(h^{*^{T}}_{i_1 i_2\ldots i_{2m-1}} - \mathbb{E}(h^{*^{T}}_{i_1 i_2\ldots i_{2m-1}})|\ X_{i_2}),$$
$$V^{T'}(i_2) =\ \mathbb{E}(h^{*^{T'}}_{i_1 i_2\ldots i_{2m-1}} - \mathbb{E}(h^{*^{T'}}_{i_1 i_2\ldots i_{2m-1}})|\ X_{i_2}),$$

where $h^{*^T}_{i_1 i_2 \ldots i_{2m-1}} = h^{(1)}_{i_1 i_2 \ldots i_m} \, h^{(m)}_{i_1 i_{m+1} \ldots i_{2m-1}}$ and $h^{*^{T'}}_{i_1 i_2 \ldots i_{2m-1}} = h^{(1)}_{i_1 i_2 \ldots i_m} \, h^{(0)}_{i_1 i_{m+1} \ldots i_{2m-1}}$.
Again observe that as $n \to \infty$

$$\begin{aligned}
&\mathbb{P}(\sum_{1 \le i_1 \ne \ldots \ne i_{2m-1} \le n} V(i_2) \ne \sum_{1 \le i_1 \ne \ldots \ne i_{2m-1} \le n} V^{T'}(i_2) \ ) \\
&\le \ \mathbb{P}(\sum_{1 \le i_1 \ne \ldots \ne i_{2m-1} \le n} V(i_2) \ne \sum_{1 \le i_1 \ne \ldots \ne i_{2m-1} \le n} V^{T}(i_2) \ ) \\
&\quad + \mathbb{P}(\sum_{1 \le i_1 \ne \ldots \ne i_{2m-1} \le n} V^{T}(i_2) \ne \sum_{1 \le i_1 \ne \ldots \ne i_{2m-1} \le n} V^{T'}(i_2) \ ) \\
&\le \ n \ \mathbb{P}(|h^{(m)}_{12\ldots m}| > n^{3/5}) + \ \mathbb{P}(|h^{(m)}_{1m+1\ldots 2m-1}| > \log(n)) \\
&\le \ \mathbb{E}[\ |h_{12\ldots m}|^{\frac{5}{3}} \ \mathbf{1}_{(|h| > n^{3/5})} \ ] + \ \mathbb{P}(|h_{1m+1\ldots 2m-1}| > \log(n)) \\
&\quad \longrightarrow 0.
\end{aligned}$$

An application of Markov inequality yields

$$\begin{aligned}
&\mathbb{P} \ (\ | \ [n]^{-2m+1} \sum_{1 \le i_1 \ne \ldots \ne i_{2m-1} \le n} V^{T'}(i_2) \ | > \epsilon) \\
&\le \ K\epsilon^{-2} \ [n - (2m-2)]^{-1} \ n \ n^{-1} \ \mathbb{E}(h^{(1)}_{12\ldots m} \ h^{(0)}_{1m+1\ldots 2m-1})^2 \\
&\le \ K\epsilon^{-2} \ [n - (2m-2)]^{-1} \ n \ n^{-\frac{3}{10}} \ \log^{7/6}(n) \ \mathbb{E}|h_{12\ldots m}|^{\frac{5}{3}} \\
&\longrightarrow 0, \ \text{as } n \to \infty.
\end{aligned}$$

This complete the proof of part (a).

In the final stage of our proofs, to prove part (b) first define $\tilde{h^*}(x) = \mathbb{E}(h_{12\ldots m} \ \mathbf{1}_{(|h| > n^{\frac{3m}{5}})} | X_1 = x)$ and write

$$\begin{aligned}
&| \frac{1}{n - 2m + 2} \sum_{i \in \{1, \ldots, n\}/\{2, \ldots, 2m-1\}}^{n} \mathbb{E}(h^*_{i2\ldots 2m-1} - \mathbb{E}(h^*_{i2\ldots 2m-1}) \ |X_i) \\
&\quad + \mathbb{E}(h^*_{12\ldots 2m-1}) - \ \frac{1}{n} \sum_{i=1}^{n} \tilde{h}_1^2(X_i) \ | \\
&= | \frac{1}{n - 2m + 2} \sum_{i \in \{1, \ldots, n\}/\{2, \ldots, 2m-1\}}^{n} \mathbb{E}(h^*_{i2\ldots 2m-1} \ |X_i) - \frac{1}{n} \sum_{i=1}^{n} \tilde{h}_1^2(X_i) \ | \\
&\le | \frac{1}{n - 2m + 2} \sum_{i \in \{1, \ldots, n\}/\{2, \ldots, 2m-1\}}^{n} \mathbb{E}(h^*_{i2\ldots 2m-1} \ |X_i) \\
&\quad - \ \frac{1}{n - 2m + 2} \sum_{i=1}^{n} \tilde{h}_1^2(X_i) \ | + \ \frac{2m-2}{n(n - 2m + 2)} \sum_{i=1}^{n} \tilde{h}_1^2(X_i) \\
&\le | \frac{1}{n - 2m + 2} \sum_{i \in \{1, \ldots, n\}/\{2, \ldots, 2m-1\}}^{n} \mathbb{E}(h^*_{i2\ldots 2m-1} \ |X_i)
\end{aligned}$$

$$
- \frac{1}{n-2m+2} \sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} \tilde{h}_1^2(X_i) \mid + \frac{1}{n-2m+2} \sum_{i=2}^{2m-1} \tilde{h}_1^2(X_i)
$$

$$
+ \frac{2m-2}{n(n-2m+2)} \sum_{i=1}^{n} \tilde{h}_1^2(X_i)
$$

$$
= \frac{1}{n-2m+2} \mid \sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} [-\tilde{h^*}(X_i)\,]\,[2\tilde{h}_1^{(m)}(X_i) + \tilde{h^*}(X_i)] \mid
$$

$$
+ \frac{1}{n-2m+2} \sum_{i=2}^{2m-1} \tilde{h}_1^2(X_i) + \frac{2m-2}{n(n-2m+2)} \sum_{i=1}^{n} \tilde{h}_1^2(X_i)
$$

$$
\leq \frac{1}{n-2m+2} [\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} \tilde{h}_1^2(X_i)]^{1/2} [\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} \tilde{h^*}^2(X_i)]^{1/2}
$$

$$
+ \frac{1}{n-2m+2} \sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} \tilde{h^*}^2(X_i) \; + \; \frac{1}{n-2m+2} \sum_{i=2}^{2m-1} \tilde{h}_1^2(X_i)
$$

$$
+ \frac{2m-2}{n(n-2m+2)} \sum_{i=1}^{n} \tilde{h}_1^2(X_i). \tag{5}
$$

It is easy to see that as $n \to \infty$, we have $\frac{1}{n-2m+2} \sum_{i=2}^{2m-1} \tilde{h}_1^2(X_i) = o_P(1)$. Also in view of Corollary A, i.e., Raikov theorem, we have $\frac{2m-2}{n(n-2m+2)} \sum_{i=1}^{n} \tilde{h}_1^2(X_i) = o_P(1)$, as $n \to \infty$. Hence, in view of (5), in order to complete the proof of part (b), it suffices to show that as $n \to \infty$,

$$
\frac{1}{n-2m+2} \sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} \tilde{h^*}^2(X_i) = o_P(1).
$$

To prove the latter relation we first use Markov inequality and conclude

$$
\begin{aligned}
&\mathbb{P}(\sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} \tilde{h^*}^2(X_i) > \; \epsilon\ (n-2m+2)\ ) \\
&\leq \; \epsilon^{-\frac{1}{2}}\ (n-2m+2)^{-\frac{1}{2}} \sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-1\}} \mathbb{E} \mid \tilde{h^*}^2(X_i) \mid^{\frac{1}{2}} \\
&\leq \; \epsilon^{-\frac{1}{2}}\ (n-2m+2)^{\frac{1}{2}}\ \mathbb{E} \mid \tilde{h^*}(X_1) \mid \\
&\leq \; \epsilon^{-\frac{1}{2}}\ (n-2m+2)^{\frac{1}{2}}\ n^{-\frac{1}{2}}\ n^{\frac{1}{2}}\ \mathbb{E}[\ |h_{12\dots m}|\ \mathbf{1}_{(|h|>n^{\frac{3m}{5}})}\ ] \\
&\leq \; \epsilon^{-\frac{1}{2}}\ (n-2m+2)^{\frac{1}{2}}\ n^{-\frac{1}{2}}\ \mathbb{E}[\ |h_{12\dots m}|^{\frac{5}{6m}+1}\ \mathbf{1}_{(|h|>n^{\frac{3m}{5}})}\ ] \\
&\quad \longrightarrow 0, \text{ as } n \to \infty.
\end{aligned}
$$

The latter relation is true since for $m \geq 2$, we have that $\frac{5}{6m} + 1 \leq \frac{5}{3}$, and this completes the proof of part (b) and those of Proposition 8 and Theorem 2.

Example. Let $X_1, X_2, \dots,$ be a sequence of i.i.d. random variables with the density function

$$f(x) = \begin{cases} |x-a|^{-3}, & |x-a| \geq 1,\ a \neq 0, \\ 0 & , \ \text{elsewhere}. \end{cases}$$

Consider the parameter $\theta = \mathbb{E}^m(X_1) = a^m$, where $m \geq 1$ is a positive integer, and the kernel $h(X_1, \ldots, X_m) = \prod_{i=1}^m X_i$. Then with m, n satisfying $n \geq m$, the corresponding U-statistic is

$$U_n = n()m^{-1} \sum_{C(n,m)} \prod_{j=1}^m X_{i_j}.$$

Simple calculation shows that $\tilde{h}_1(X_1) = X_1 \ a^{m-1} \ - \ a^m$.

It is easy to check that $\mathbb{E}|h(X_1, \ldots, X_m)|^{\frac{5}{3}} < \infty$ and that $\tilde{h}_1(X_1) \in DAN$ (cf. Gut, [**?**]).

For the pseudo-selfnormalized process

$$U^*_{[nt]} = \begin{cases} 0, & 0 \leq t < \dfrac{m}{n}, \\ \dfrac{[nt]()m^{-1} \sum_{C([nt],m)} \prod_{j=1}^m X_{i_j} \ - \ a^m}{(\sum_{i=1}^n (X_i a^{m-1} \ - \ a^m)^2)^{\frac{1}{2}}}, & \dfrac{m}{n} \leq t \leq 1. \end{cases}$$

Nasari in [**?**] concludes that $\frac{[nt]}{m} \ U^*_{[nt]} \longrightarrow_d W(t)$ on $(D[0,1], \rho)$, where ρ is the sup-norm for functions in $D[0,1]$ and $\{W(t), 0 \leq t \leq 1\}$ is a standard Wiener process.

The studentized U-process based on U_n here is defined as follows.

$$U^{stu}_{[nt]} = \begin{cases} 0, & 0 \leq t < \dfrac{m}{n}, \\ \dfrac{[nt]()m^{-1} \sum_{C([nt],m)} \prod_{j=1}^m X_{i_j} - \theta}{\sqrt{(n-1)\sum_{i=1}^n (U^i_{n-1} - U_n)^2}}, & \dfrac{m}{n} \leq t \leq 1, \end{cases}$$

where, by (2),

$$(n-1) \sum_{i=1}^n (U^i_{n-1} - U_n)^2 = \frac{m^2(n-1)}{(n-m)^2} \Big\{ \sum_{i=1}^n X_i^2 [n-1()m-1^{-1} \sum_{\substack{1 \leq i_2 < \ldots < i_m \leq n \\ i_2, \ldots, i_m \neq i}} \prod_{j=2}^m X_{i_j}]^2$$

$$- \ n \ [n()m^{-1} \sum_{C(n,m)} \prod_{j=1}^m X_{i_j}]^2 \Big\}.$$

In view of $U^{stu}_{[nt]}$ and $U^*_{[nt]}$, our Main Theorem is applicable for $U^{stu}_{[nt]}$ provided Theorem 1 continues hold true in this case. Hence, part (c) of Main Theorem implies that $[nt] \ U^{stu}_{[nt]} \rightarrow_d \ W(t)$ on $(D[0,1],\rho)$, where ρ is the sup-norm for functions in $D[0,1]$ and $\{W(t), 0 \leq t \leq 1\}$ is a standard Wiener process.

Appendix: Proof of Proposition 4

As it was mentioned before, the proof of this proposition can be done by modifying that of Theorem 2, except for that some of the steps are not required. This is due to the presence of the extra term of n with negative power i.e., n^{-j+1} in this proposition, where $j = 2, \dots, m-1$, and $m \geq 3$. It is clear that among the statistics in proposition 4 the one associated to $j = 2$ has the largest extra term of n^{-1}. Hence, we shall only show that as $n \to \infty$,

$$[n]^{-2m+1} \sum_{1 \leq i_1 \neq \dots \neq i_{2m-2} \leq n} h(X_{i_1}, X_{i_2}, X_{i_3}, \dots, X_{i_m}) \times h(X_{i_1}, X_{i_2}, X_{i_{m+1}}, \dots, X_{i_{2m-2}}) = o_P(1). \qquad \text{(I)}$$

To prove (I), consider the following setup:

$$\begin{aligned}
h_{1\dots m} &:= h(X_1, \dots, X_m), \\
h^{(m)}_{1\dots m} &:= h_{1\dots m}\, \mathbf{1}_{(|h| \leq n^{\frac{3m}{5}})}, \\
h^{**}_{12\dots 2m-2} &:= h^{(m)}_{123\dots m}\, h^{(m)}_{12\ m+1\dots 2m-2}, \\
h^{(j)}_{1\dots m} &:= h^{(m)}_{1\dots m}\, \mathbf{1}_{(|h^{(m)}| \leq n^{\frac{3j}{5}})},\ j = 1, \dots, m-1, \\
h^{(0)}_{1\dots m} &:= h^{(m)}_{1\dots m}\, \mathbf{1}_{(|h^{(m)}| \leq \log(n))},
\end{aligned}$$

where $\mathbf{1}_A$ is the indicator function of the set A.

Now observe that as $n \to \infty$

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{1 \leq i_1 \neq \dots \neq i_{2m-2} \leq n} h_{i_1 i_2 i_3 \dots i_m}\, h_{i_1 i_2 i_{m+1} \dots i_{2m-2}} \neq \sum_{1 \leq i_1 \neq \dots \neq i_{2m-2} \leq n} h^{(m)}_{i_1 i_2 i_3 \dots i_m}\, h^{(m)}_{i_1 i_2 i_{m+1} \dots i_{2m-2}} \Big) \\
&\leq\ n^m\, \mathbb{P}\,(\, |h_{1\dots m}| >\ n^{\frac{3m}{5}}\ ) \\
&\leq \mathbb{E}\,[\, |h_{1\dots m}|^{\frac{5}{3}}\, 1_{(|h_{1\dots m}| > n^{\frac{3m}{5}})}\,] \longrightarrow 0.
\end{aligned}$$

In view of the latter asymptotic equivalency and our setup, in order to prove (I), we need to show that as $n \to \infty$,

$$[n]^{-2m+1} \sum_{1 \leq i_1 \neq \dots \neq i_{2m-2} \leq n} h^{**}_{i_1, i_2, \dots, i_{2m-2}} = o_P(1).$$

Similarly to what we had in the proof of Theorem 2 we write

$$\begin{aligned}
&\sum_{1 \leq i_1 \neq \dots \neq i_{2m-2} \leq n} h^{**}_{i_1 \dots i_{2m-2}} \\
&= \sum_{1 \leq i_1 \neq \dots \neq i_{2m-2} \leq n} \Big\{ \sum_{d=1}^{2m-2} (-1)^{2m-2-d} \sum_{1 \leq j_1 < \dots < j_d \leq 2m-2} \mathbb{E}(h^{**}_{i_1 \dots i_{2m-2}} - \mathbb{E}(h^{**}_{i_1 \dots i_{2m-2}}) |\, X_{i_{j_1}}, \dots X_{i_{j_d}})
\end{aligned}$$

$$+\sum_{c=1}^{2m-3}\sum_{1\le k_1<\ldots<k_c\le 2m-2}\sum_{d=1}^{c}(-1)^{c-d}\sum_{1\le j_1<\ldots<j_d\le c}\mathbb{E}(h^{**}_{i_1\ldots i_{2m-1}}-\mathbb{E}(h^{**}_{i_1\ldots i_{2m-2}})|\, X_{i_{k_{j_1}}},\ldots X_{i_{k_{j_d}}})$$

$$+\,\mathbb{E}\,(h^{**}_{i_1\ldots i_{2m-2}})\}$$

$$:=\sum_{1\le i_1\neq\ldots\neq i_{2m-2}\le n}V^*(i_1,\ldots,i_{2m-2})+\sum_{c=1}^{2m-3}\sum_{1\le k_1<\ldots<k_c\le 2m-2}\sum_{1\le i_1\neq\ldots\neq i_{2m-2}\le n}V^*(i_{k_1},\ldots,i_{k_c})$$

$$+\ \textstyle\sum_{1\le i_1\neq\ldots\neq i_{2m-2}\le n}\mathbb{E}(h^{**}_{i_1\ldots i_{2m-2}}).$$

To prove (I), we shall show the asymptotic negligibility of all of the above terms in the next three propositions.

Proposition 4.1: *If* $\mathbb{E}\,|h_{1\ldots m}|^{\frac{5}{3}}<\infty$, *then, as* $n\to\infty$

$$[n]^{-2m+1}\sum_{1\le i_1\neq\ldots\neq i_{2m-2}\le n}V^*(i_1,\ldots,i_{2m-2})\ =o_P(1).$$

Proof of Proposition 4.1

For throughout use K will be a positive constant that may be different at each stage.

Since $V^*(i_1,\ldots,i_{2m-2})$ posses the property of *degeneracy* we can apply Lemma 1 following a Markov inequality for the associated statistic and write, for $\epsilon>0$,

$$\mathbb{P}\ (\ |\ [n]^{-2m+2}\sum_{1\le i_1\neq\ldots\neq i_{2m-2}\le n}V^*(i_1,\ldots,i_{2m-2})|>\epsilon\ (n-2m+2)\ )$$

$$\le\epsilon^{-2}\ \ (n-2m+2)^{-2}\ \mathbb{E}\ [\ [n]^{-2m+2}\sum_{1\le i_1\neq\ldots\neq i_{2m-1}\le n}V^*(i_1,\ldots,i_{2m-2})\ ]^2$$

$$\le\epsilon^{-2}\ \ (n-2m+2)^{-2}\ [n]^{-2m+2}\ \mathbb{E}\ [\ V^*(1,\ldots,2m-2)\ ]^2$$

$$\le\ K\,\epsilon^{-2}\,(n-2m+2)^{-2}\,[n]^{-2m+2}\,n^{2m}\,n^{-2m}\,\mathbb{E}\,[\,h^{(m)}_{12\ 3\ldots m}\,h^{(m)}_{12\ m+1\ldots 2m-2}\,]^2$$

$$\le\ K\,\epsilon^{-2}\,(n-2m+2)^{-2}\,[n]^{-2m+2}\,n^{2m}\,n^{-2m}\,n^{\frac{7m}{5}}\,\mathbb{E}\,|\,h_{12\ldots m}\,|^{\frac{5}{3}}$$

$$\longrightarrow 0,\ \text{as}\ n\to\infty.$$

Proposition 4.2. *If* $\mathbb{E}\,|h_{1\ldots m}|^{\frac{5}{3}}<\infty$, *then, as* $n\to\infty$,

$$[n]^{-2m+1}\sum_{1\le i_1\neq\ldots\neq i_{2m-2}\le n}V(i_{k_1},\ldots,i_{k_c})\ =o_P(1),$$

where $c=2,\ldots,2m-3$ *and* $1\le k_1<\ldots<k_c\le 2m-2$.

Proof of Proposition 4.2

The proof will be stated in three cases according to the values of k_1 and k_2 as follows.

Case $k_1 = 1$ *and* $k_2 = 2$
Let s and t be respectively the number of elements of the sets $\{i_{k_1}, \ldots, i_{k_c}\} \cap \{i_1, i_2, i_3, \ldots, i_m\}$ and $\{i_{k_1}, \ldots, i_{k_c}\} \cap \{i_1, i_2, i_{m+1}, \ldots, i_{2m-1}\}$. It is clear that in this case, i.e., $k_1 = 1$ and $k_2 = 2$, we have that $s, t \geq 2$ and $s + t = c + 2$. Now define

$$V^{*^T}(i_{k_1}, \ldots, i_{k_c}) = \sum_{d=1}^{c} (-1)^{c-d} \sum_{1 \leq j_1 < \ldots < j_d \leq c} \mathbb{E}(h^{**^T}_{i_1 \ldots i_{2m-2}} - \mathbb{E}(h^{**^T}_{i_1 \ldots i_{2m-2}}) \mid X_{i_{k_{j_1}}}, \ldots, X_{i_{k_{j_d}}}),$$

$$V^{*T'}(i_{k_1}, \ldots, i_{k_c}) = \sum_{d=1}^{c} (-1)^{c-d} \sum_{1 \leq j_1 < \ldots < j_d \leq c} \mathbb{E}(h^{**^{T'}}_{i_1 \ldots i_{2m-2}} - \mathbb{E}(h^{**^{T'}}_{i_1 \ldots i_{2m-2}}) \mid X_{i_{k_{j_1}}}, \ldots, X_{i_{k_{j_d}}}),$$

where $h^{**^T}_{i_1 \ldots i_{2m-2}} = h^{(s)}_{i_1 i_2 \ i_3 \ldots i_m} \ h^{(m)}_{i_1 i_2 \ i_{m+1} \ldots i_{2m-2}}$ and $h^{**^{T'}}_{i_1 \ldots i_{2m-2}} = h^{(s)}_{i_1 i_2 \ i_3 \ldots i_m} \ h^{(t)}_{i_1 i_2 \ i_{m+1} \ldots i_{2m-2}}$.

Now observe that as $n \to \infty$

$$\mathbb{P}\,\Big(\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-2} \leq n} V^*(i_{k_1}, \ldots, i_{k_c}) \ \neq \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-2} \leq n} V^{*^{T'}}(i_{k_1}, \ldots, i_{k_c}) \Big)$$

$$\leq \mathbb{P}\,\Big(\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-2} \leq n} V^*(i_{k_1}, \ldots, i_{k_c}) \ \neq \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-2} \leq n} V^{*^T}(i_{k_1}, \ldots, i_{k_c}) \Big)$$

$$+ \mathbb{P}\,\Big(\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-2} \leq n} V^{*^T}(i_{k_1}, \ldots, i_{k_c}) \ \neq \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-2} \leq n} V^{*^{T'}}(i_{k_1}, \ldots, i_{k_c}) \Big)$$

$$\leq \ n^s \ \mathbb{P}\ (\ |h^{(m)}_{12 \ 3 \ldots m}| > \ n^{\frac{3s}{5}}\ ) + \ n^t \ \mathbb{P}\ (\ |h^{(m)}_{12 \ m+1 \ldots 2m-1}| > \ n^{\frac{3t}{5}}\ )$$

$$\leq \ \mathbb{E}\ [\ |h_{12 \ 3 \ldots m}|^{\frac{5}{3}} \ \mathbf{1}_{(|h| > n^{\frac{3s}{5}})}\ ] + \ \mathbb{E}\ [\ |h_{12 \ m+1 \ldots 2m-2}|^{\frac{5}{3}} \ \mathbf{1}_{(|h| > n^{\frac{3t}{5}})}\ ] \longrightarrow 0.$$

The latter relation suggests that $\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-2} \leq n} V^*(i_{k_1}, \ldots, i_{k_c})$ and $\sum_{1 \leq i_1 \neq \ldots \neq i_{2m-2} \leq n} V^{*^{T'}}(i_{k_1}, \ldots, i_{k_c})$ are asymptotically equivalent in probability.

Since $V^{*^{T'}}(i_{k_1}, \ldots, i_{k_c})$ is *degenerate*, Markov inequality followed by an application of Lemma 1 yields,

$$\mathbb{P}\ (\ |\ [n]^{-2m+2} \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-2} \leq n} V^{*^{T'}}(i_{k_1}, \ldots, i_{k_c})\ | \ > \epsilon\ (n - 2m + 2)\ )$$

$$\leq \ \epsilon^{-2}\ (n - 2m + 2)^{-2}\ \mathbb{E}\ [\ [n]^{-2m+2} \sum_{1 \leq i_1 \neq \ldots \neq i_{2m-2} \leq n} V^{*^{T'}}(i_{k_1}, \ldots, i_{k_c})\ ]^2$$

$$\leq\ K\epsilon^{-2}\,(n{-}2m{+}2)^{-2}\,[n{-}(2m{-}2{-}c)]^{-c}\,\mathbb{E}\,[\,h^{(s)}_{123\ldots m}\,h^{(t)}_{12\ m+1\ldots 2m-1}\,]^2$$

$$\leq\ K\epsilon^{-2}\,(n{-}2m{+}2)^{-2}\,[n{-}(2m{-}1{-}c)]^{-c}\,n^{c+2}\,n^{-c-2}\,n^{\frac{7(t+s)}{10}}\,\mathbb{E}\,|\,h_{1\ldots m}\,|^{\frac{5}{3}}$$

$$\longrightarrow 0,\ \text{as } n \to \infty.$$

This completes the proof of Proposition 4.2 when $k_1 = 1$ and $k_2 = 2$.

Case either $k_1 \neq 1$ *or* $k_2 \neq 2$

Let s, t be as were defined in the previous case and note that here we have that $s, t \geq 1$ and $s + t = c + 1$. The proof of Proposition 4.2 in this case results from a similar argument to what was given for the previous case, hence the details are omitted.

Case $k_1 \neq 1, k_2 \neq 2$

Let s, t be as what were defined in the previous two cases and note that in this case we have $s, t \geq 0$ and $s + t = c$. Also let V^{*^T} and $V^{*^{T'}}$ as they were defined in the case $k_1 = 1, k_2 = 2$ and observe that as $n \to \infty$ we have

$$\mathbb{P}\,\Big(\sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^*(i_{k_1},\ldots,i_{k_c})\ \neq \sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^{*^{T'}}(i_{k_1},\ldots,i_{k_c})\ \Big)$$

$$\leq \mathbb{P}\,\Big(\sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^*(i_{k_1},\ldots,i_{k_c})\ \neq \sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^{*^{T}}(i_{k_1},\ldots,i_{k_c})\ \Big)$$
$$+\,\mathbb{P}\,\Big(\sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^{*^T}(i_{k_1},\ldots,i_{k_c})\ \neq \sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^{*^{T'}}(i_{k_1},\ldots,i_{k_c})\Big)$$

$$\leq \begin{cases} n^s\,\mathbb{P}\,(\,|h^{(m)}_{12\ 3\ldots m}| >\ n^{\frac{3s}{5}}\,) +\ n^t\,\mathbb{P}\,(\,|h^{(m)}_{12\ m+1\ldots 2m-2}| >\ n^{\frac{3t}{5}}\,), & \text{s,t} > 0,\ \text{s+t=c};\\ n^c\,\mathbb{P}(\,|h^{(m)}_{12\ 3\ldots m}| >\ n^{\frac{3c}{5}}\,) +\ \mathbb{P}\,(\,|h^{(m)}_{12\ m+1\ldots 2m-2}| >\ \log(n)\,), & \text{s=c,t=0} \end{cases}$$

$$\leq \begin{cases} \mathbb{E}\,[\,|h_{12\ 3\ldots m}|^{\frac{5}{3}}\,\mathbf{1}_{(|h|>n^{\frac{3s}{5}})}\,] +\ \mathbb{E}\,[\,|h_{12\ m+1\ldots 2m-2}|^{\frac{5}{3}}\,\mathbf{1}_{(|h|>n^{\frac{3t}{5}})}\,], & \text{s,t} > 0,\ \text{s+t=c};\\ \mathbb{E}\,[\,|h_{12\ldots m}|^{\frac{5}{3}}\,\mathbf{1}_{(|h|>n^{\frac{3c}{5}})}\,] +\ \mathbb{P}\,(\,|h^{(m)}_{1m+1\ldots 2m-2}| >\ \log(n)\,), & \text{s=c,t=0} \end{cases}$$

$$\longrightarrow 0.$$

Applying Markov inequality followed by an application Lemma 1 once again yields

$$\mathbb{P}\,(\,|\,[n]^{-2m+2}\sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq} V^{*T'}(i_{k_1},\ldots,i_{k_c})\,| > \epsilon\,n\,(n{-}2m{+}2)\,)$$

$$\leq K\epsilon^{-2}\,(n{-}2m{+}2)^{-2}\,[n{-}(2m{-}2{-}c)]^{-c}\,n^{c+2}\,n^{-c-2}\,\mathbb{E}\,[\,h^{(s)}_{123\ldots m}\,h^{(t)}_{12\ m+1\ldots 2m-2}\,]^2$$

$$\leq \begin{cases} K\epsilon^{-2}\ (n-2m+2)^{-2}\ [n-(2m-2-c)]^{-c}\ n^{c+2}\ n^{-c-2}\ n^{\frac{7c}{10}}\ \mathbb{E}|h_{12\ldots m}|^{\frac{5}{3}}, & \text{s,t} > 0,\ \text{s+t=c;} \\ K\epsilon^{-2}\ (n-2m+2)^{-2}\ [n-(2m-2-c)]^{-c}\ n^{c+2}\ n^{-c-2}\ n^{\frac{7c}{10}}\ \log^{\frac{7}{6}}(n)\ \mathbb{E}|h_{12\ldots m}|^{\frac{5}{3}}, & \text{s=c,t=0} \end{cases}$$

$\longrightarrow 0$, as $n \to \infty$.

This completes the proof of Proposition 4.2.

As the last step of the proof of Proposition 4, in the next result we deal with terms of the form of sums of i.i.d. random variables (cf. Remark 7).

Proposition 4.3. *If* $\mathbb{E}|h_{1\ldots m}|^{\frac{5}{3}} < \infty$, *then, as* $n \to \infty$

(a) $[n]^{-2m+1} \sum_{1\leq\neq i_1\neq\ldots\neq i_{2m-2}\leq n} V^*(i_{k_1}) = o_P(1),\ k_1 \in \{3,\ldots,2m-2\}$,

$$(b)\ \frac{1}{(n-2m+2)(n-2m+3)} \sum_{i\in\{1,\ldots,n\}/\{1,3,\ldots,2m-2\}}^{n} \mathbb{E}(h^{**}_{1i3\ldots 2m-2} - \mathbb{E}(h^{**}_{1i3\ldots 2m-2})\,|X_i) = o_P(1),$$

$$(c)\frac{1}{(n-2m+2)(n-2m+3)} \sum_{i\in\{1,\ldots,n\}/\{2,\ldots,2m-2\}}^{n} \mathbb{E}(h^{**}_{i2\ldots 2m-2} - \mathbb{E}(h^{**}_{i2\ldots 2m-2})\,|X_i)$$

$$+\ \frac{1}{n-2m+2}\ \mathbb{E}(h^{**}_{12\ldots 2m-2})\ = o_P(1).$$

Proof of Proposition 4.3

First we give the proof of part (a). Due to similarities, we shall state the proof only for the case that $k_1 = 3$.

Define

$$V^{*^T}(i_3) = \ \mathbb{E}(h^{**^T}_{i_1i_2\ldots i_{2m-2}} - \mathbb{E}(h^{**^T}_{i_1i_2\ldots i_{2m-2}})|\ X_{i_3}),$$
$$V^{*^{T'}}(i_3) = \ \mathbb{E}(h^{**^{T'}}_{i_1i_2\ldots i_{2m-2}} - \mathbb{E}(h^{**^{T'}}_{i_1i_2\ldots i_{2m-2}})|\ X_{i_3}),$$

where $h^{**^T}_{i_1i_2\ldots i_{2m-2}} = h^{(1)}_{i_1i_2i_3\ldots i_m}\ h^{(m)}_{i_1i_2i_{m+1}\ldots i_{2m-2}}$ and $h^{**^{T'}}_{i_1i_2\ldots i_{2m-2}} = h^{(1)}_{i_1i_2i_3\ldots i_m}\ h^{(0)}_{i_1i_2i_{m+1}\ldots i_{2m-2}}$. Again observe that as $n \to \infty$

$$\mathbb{P}\Big(\sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^*(i_3) \neq \sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^{*^{T'}}(i_3)\Big)$$

$$\leq\ \mathbb{P}\Big(\sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^*(i_3) \neq \sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^{*^T}(i_3)\Big)$$

$$+\ \mathbb{P}\Big(\sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^{*^T}(i_3) \neq \sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^{*^{T'}}(i_3)\Big)$$

$$\leq \; n\,\mathbb{P}(|h^{(m)}_{123\ldots m}| > n^{3/5}) + \mathbb{P}(|h^{(m)}_{12\;m+1\ldots 2m-2}| > \log(n))$$
$$\longrightarrow 0.$$

Applying Markov inequality we arrive at

$$\mathbb{P}(\;|\frac{1}{n-2m+3}\sum_{1\leq i_1\neq\ldots\neq i_{2m-2}\leq n} V^{*^{T'}}(i_3)\;| > \;\epsilon\;(n-2m+2)\;)$$

$$\leq \; K\;\epsilon^{-2}\;(n-2m+2)^{-2}\;(n-2m+3)^{-1}\;\mathbb{E}(h^{(1)}_{123\ldots m}h^{(0)}_{12\;m+1\ldots 2m-2})^2$$
$$\longrightarrow 0,\;\text{as } n\to\infty.$$

This completes the proof of part (a).

Next to prove part (b) define

$$h^{**^{T}} = \; h^{(1)}_{123\ldots m}h^{(m)}_{12\;m+1\ldots 2m-2},$$
$$h^{**^{T'}} = \; h^{(1)}_{123\ldots m}h^{(1)}_{12\;m+1\ldots 2m-2},$$

and observe that as $n\to\infty$ we have

$$\mathbb{P}\Big(\sum_{i\in\{1,\ldots,n\}/\{1,3,\ldots,2m-2\}}^{n}\mathbb{E}(h^{**}_{1i3\ldots 2m-2}-\mathbb{E}(h^{**}_{1i3\ldots 2m-2})\,|X_i)$$
$$\neq \sum_{i\in\{1,\ldots,n\}/\{1,3,\ldots,2m-2\}}^{n}\mathbb{E}(h^{**^{T'}}_{1i3\ldots 2m-2}-\mathbb{E}(h^{**^{T'}}_{1i3\ldots 2m-2})\,|X_i)\Big)$$

$$\leq \mathbb{P}\Big(\sum_{i\in\{1,\ldots,n\}/\{1,3,\ldots,2m-2\}}^{n}\mathbb{E}(h^{**}_{1i3\ldots 2m-2}-\mathbb{E}(h^{**}_{1i3\ldots 2m-2})\,|X_i)$$
$$\neq \sum_{i\in\{1,\ldots,n\}/\{1,3,\ldots,2m-2\}}^{n}\mathbb{E}(h^{**^{T}}_{1i3\ldots 2m-2}-\mathbb{E}(h^{**^{T}}_{1i3\ldots 2m-2})\,|X_i)\Big)$$
$$+\,\mathbb{P}\Big(\sum_{i\in\{1,\ldots,n\}/\{1,3,\ldots,2m-2\}}^{n}\mathbb{E}(h^{**^{T}}_{1i3\ldots 2m-2}-\mathbb{E}(h^{**^{T}}_{1i3\ldots 2m-2})\,|X_i)$$
$$\neq \sum_{i\in\{1,\ldots,n\}/\{1,3,\ldots,2m-2\}}^{n}\mathbb{E}(h^{**^{T'}}_{1i3\ldots 2m-2}-\mathbb{E}(h^{**^{T'}}_{1i3\ldots 2m-2})\,|X_i)\Big)$$
$$\leq n\;\mathbb{P}(|h^{(m)}_{123\ldots m}| > n^{3/5}) + \;n\;\mathbb{P}(|h^{(m)}_{12\;m+1\ldots 2m-2}| > n^{3/5})$$
$$\longrightarrow 0.$$

Hence another application of Markov inequality yields,

$$\mathbb{P}\Big(\;|\frac{1}{n-2m+3}\sum_{i\in\{1,\ldots,n\}/\{1,3,\ldots,2m-2\}}^{n}\mathbb{E}(h^{**^{T'}}_{1i3\ldots 2m-2}-\mathbb{E}(h^{**^{T'}}_{1i3\ldots 2m-2})\,|X_i)\;| > \epsilon\;(n-2m+2)\Big)$$

$$\leq \; K\;\epsilon^{-2}\;(n-2m+2)^{-2}\;(n-2m+3)^{-1}\;\mathbb{E}(h^{(1)}_{123\ldots m}h^{(1)}_{12\;m+1\ldots 2m-2})^2$$

$\longrightarrow 0, \text{ as } n \to \infty.$
Now the proof of part (b) is complete.

To prove part (c) we only need to observe that

$$\frac{1}{(n-2m+2)(n-2m+3)} \sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-2\}}^{n} \mathbb{E}(h^{**}_{i2\dots 2m-2} - \mathbb{E}(h^{**}_{i2\dots 2m-2})\,|X_i)$$

$$+\frac{1}{n-2m+2}\,\mathbb{E}(h^{**}_{12\dots 2m-2}) = \frac{1}{(n-2m+2)(n-2m+3)} \sum_{i\in\{1,\dots,n\}/\{2,\dots,2m-2\}}^{n} \mathbb{E}(h^{**}_{i2\dots 2m-2}\,|X_i).$$

The rest of the proof is similar to that of part (b), hence the details are omitted. Now the proof of Proposition 4.3 and that of (I) is complete.

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