Asymptotics of Studentized U-type processes for changepoint problems

Miklós Csörgő
Carleton University, Ottawa, Canada

Barbara Szyszkowicz
Carleton University, Ottawa, Canada

Qiying Wang
University of Sydney, Australia

ABSTRACT

This paper investigates weighted approximations for studentized U-statistics type processes, both with symmetric and antisymmetric kernels, only under the assumption that the distribution of the projection variate is in the domain of attraction of the normal law. The results can be used for testing the null assumption of having a random sample versus the alternative that there is a change in distribution in the sequence.

Key Words and Phrases: Weighted approximation, U-statistics type process, Studentization, Change-point, Symmetric and antisymmetric kernels, Gaussian processes.

AMS 1991 Subject Classification: Primary 60F05, 60F17, Secondary 62E20.

Running Head: Asymptotics of Studentized U-type processes.

Csörgő and Szyszkowicz acknowledge partial research support from NSERC Canada Grants at Carleton University, Ottawa and Wang acknowledges partial research support from Australian Research Council.

1 Introduction and main results: the case of symmetric kernels

Let $X, X_1, X_2, ...$ be a sequence of non-degenerate i.i.d. random variables with distribution function F. Suppose we are interested in testing the null hypothesis:

$$H_0: \quad X_i,\ 1 \le i \le n,\ \textit{have the same distribution},$$

against the one change alternative:

$$\begin{aligned} H_A: \quad & \textit{there is an integer } k,\ 1 \le k < n,\ \textit{such that} \\ & P(X_1 \le t) = \cdots = P(X_k \le t),\ P(X_{k+1} \le t) = \cdots = P(X_n \le t) \\ & \textit{for all } t \textit{ and } P(X_k \le t_0) \ne P(X_{k+1} \le t_0) \textit{ for some } t_0. \end{aligned}$$

Testing for this kind of a change in distribution has been studied considerably in the literature by using parametric as well as non-parametric methods. One of the non-parametric methods was proposed by Csörgő and Horváth (1988a, b), who used functionals of a $U-$ statistics type (U-type, from now on) process to test H_0 against H_A. Let $h(x, y)$ be a measurable real valued symmetric function. The $U-$type process of Csörgő and Horváth (1988a, b) is defined by

$$U_n(t) = Z_{[(n+1)t]} - n^2 t(1-t)\theta, \qquad 0 \le t \le 1,$$

where $\theta = Eh(X_1, X_2)$, and

$$Z_k = \sum_{i=1}^{k} \sum_{j=k+1}^{n} h(X_i, X_j), \qquad 1 \le k \le n.$$

While Z_k itself is not a U-statistic, it can be written as the sums of three $U-$ statistics [cf. Csörgő and Horváth (1988a, b, 1997)]. Typical choices of symmetric kernel h are xy, $(x-y)^2/2$ (the sample variance), $|x-y|$ (Gini's mean difference), and $sign(x+y)$ (Wilcoxon's one-sample statistic).

Throughout the paper, we write $g(t) = E\left(h(X,t) - \theta\right)$, $\sigma^2 = Eg^2(X_1)$ and, for later use, we define a Gaussian process Γ by

$$\Gamma(t) = (1-t)\,W(t) \,+\, t\,\left[W(1) - W(t)\right], \quad 0 \le t \le 1, \tag{1}$$

where $\{W(t), 0 \leq t < \infty\}$ is a standard Wiener process. Furthermore, let Q be the class of positive functions q on $(0,1)$, i.e., $\inf_{\delta \leq t \leq 1-\delta} q(t) > 0$ for $0 < \delta < 1$, which are nondecreasing in a neighbourhood of zero and nonincreasing in a neighbourhood of one, and let

$$I(q,c) = \int_{0+}^{1-} \frac{1}{t(1-t)} \exp\Big(-\frac{cq^2(t)}{t(1-t)}\Big) dt, \qquad 0 < c < \infty.$$

In terms of these notations, Csörgő and Horváth (1988a, b), Szyszkowicz (1991, 1992) established the following result [cf. Theorem 2.4.2 in Csörgő and Horváth (1997)].

Theorem A. *Assume H_0, $0 < \sigma^2 < \infty$ and $E|h(X_1, X_2)|^2 < \infty$. Then, on an appropriate probability space for $X, X_1, X_2, \cdots$, we can define a sequence of Gaussian processes $\{\Gamma_n(t), 0 \leq t \leq 1\}$ such that*

$$\{\Gamma_n(t), 0 \leq t \leq 1\} \stackrel{D}{=} \{\Gamma(t), 0 \leq t \leq 1\}, \tag{2}$$

for each $n \geq 1$, and as $n \to \infty$,

$$\sup_{0<t<1} \Big| n^{-3/2}\sigma^{-1} U_n(t) - \Gamma_n(t) \Big| \Big/ q(t) \;\; = \;\; o_P(1). \tag{3}$$

if and only if $I(q,c) < \infty$ for any $c > 0$.

This theorem provides a basic tool for investigating the asymptotic behaviour of many test statistics via corresponding functionals of $\Gamma(.)/q(.)$ for appropriate choices of the kernel $h(x, y)$. This, in turn, motivates the establishment of our first result, in which we reduce the moment conditions related to the kernel $h(x, y)$. Our first result reads as follows.

Theorem 1 *Assume H_0, $0 < \sigma^2 < \infty$ and $E|h(X_1, X_2)|^{4/3} < \infty$. Then, on an appropriate probability space for $X, X_1, X_2, \cdots$, we can define a sequence of Gaussian processes $\{\Gamma_n(t), 0 \leq t \leq 1\}$ such that (2) holds true, and if $I(q,c) < \infty$ for some $c > 0$, then as $n \to \infty$,*

$$\sup_{1/n \leq t \leq (n-1)/n} \Big| n^{-3/2}\sigma^{-1} U_n(t) - \Gamma_n(t) \Big| \Big/ q(t) \;\; = \;\; o_P(1). \tag{4}$$

In addition to reducing the moment conditions required in Theorem A, the result (4) of Theorem 1 generalizes (3) as well. Namely, as a direct consequence of Theorem 1, we have the following corollary.

Corollary 1 *Assume H_0, $0 < \sigma^2 < \infty$ and $E|h(X_1, X_2)|^{4/3} < \infty$. If $q \in Q$, then*

(a) we still have the conclusion of Theorem A, i.e., (3) holds true if and only if $I(q, c) < \infty$ for any $c > 0$;

(b) as $n \to \infty$,

$$n^{-3/2}\sigma^{-1}U_n(t) \Big/ q(t) \quad \Rightarrow \quad \Gamma(t) \Big/ q(t) \tag{5}$$

on $(D[0,1], \rho)$, where ρ is the sup-norm metric for functions in $D[0,1]$, if and only if $I(q, c) < \infty$ for any $c > 0$;

(c) as $n \to \infty$,

$$n^{-3/2}\sigma^{-1} \sup_{0<t<1} |U_n(t)| \Big/ q(t) \quad \to_D \quad \sup_{0<t<1} |\Gamma(t)|/q(t) \tag{6}$$

if and only if $I(q, c) < \infty$ for some $c > 0$.

When θ and σ are known, large values of the statistic on the left hand sides in (6), for example, indicate a change in the distribution, and hence, based on Corollary 1, rejection of H_0 can be quantified accordingly. Otherwise θ and σ need to be estimated. A natural estimate of θ is

$$\hat{\theta} = \frac{1}{n(n-1)} \sum_{1 \le i \ne j \le n} h(X_i, X_j),$$

and that of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{j=1}^{n} \Big(g(X_j) - \frac{1}{n} \sum_{j=1}^{n} g(X_j) \Big)^2.$$

According to the definition of $g(x)$, $g(X_j)$ still depends on the usually unknown distribution function F of X, and hence it then can not be computed explicitly. Since we have that $g(x) + \theta = \int h(x, y) dF(y)$, we can replace F by the empirical distribution function F_n of $X_1, X_2, ..., X_n$ under H_0. Consequently, we may for example estimate σ^2 by

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{j=1}^{n} \Big(\frac{1}{n-1} \sum_{\substack{i=1 \\ i \ne j}}^{n} h(X_i, X_j) - \frac{1}{n(n-1)} \sum_{1 \le i \ne j \le n} h(X_i, X_j) \Big)^2.$$

We note that this estimate is in fact the jackknife estimator of $Var(\hat{\theta})$. Now we may introduce a studentized U-type process as follows:

$$\hat{U_n}(t) = n^{-3/2}(\hat{\sigma})^{-1} \left(Z_{[(n+1)t]} - n^2 t(1-t)\hat{\theta} \right), \qquad 0 \le t \le 1.$$

This process does not depend on the unknown parameters θ and σ and we now state the following main result of this paper.

Theorem 2 *Let $q \in Q$. Assume H_0, $E|h(X_1, X_2)|^{5/3} < \infty$ and that $g(X_1)$ is in the domain of attraction of the normal law. Then, on an appropriate probability space for $X, X_1, X_2, \cdots$, we can define a sequence of Gaussian processes $\{\Gamma_n(t), 0 \le t \le 1\}$ such that (2) holds true and, as $n \to \infty$,*

$$\sup_{0<t<1} \left| \hat{U_n}(t) - \Gamma_n(t) \right| \Big/ q(t) \quad = \quad o_P(1), \tag{7}$$

if and only if $I(q, c) < \infty$ for any $c > 0$. Consequently, as $n \to \infty$,

$$\hat{U_n}(t)/q(t) \quad \Rightarrow \quad \Gamma(t) \Big/ q(t), \quad \text{on } (D[0,1], \rho) \tag{8}$$

if and only if $I(q, c) < \infty$ for any $c > 0$. Furthermore we also have

$$\sup_{0<t<1} |\hat{U_n}(t)| \Big/ q(t) \quad \to_D \quad \sup_{0<t<1} |\Gamma(t)|/q(t) \tag{9}$$

if and only if $I(q, c) < \infty$ for some $c > 0$.

Remark 1. It is interesting and also of interest to note that the class of the weight functions in (9) is bigger than that in (8) [also compare (6) with (5)]. Such a phenomenon was first noticed and proved for weighted empirical and quantile processes by Csörgő, Csörgő, Horváth and Mason [CsCsHM] (1986) and then by Csörgő and Horváth (1988b) for partial sums on assuming $E|X|^v < \infty$ for some $v > 2$. For more details along these lines, we refer to Szyszkowicz (1991, 1996, 1997), and to Csörgő, Norvaiša and Szyszkowicz (1999).

Remark 2. The condition that $0 < \sigma^2 = Eg^2(X_1) < \infty$ is the so-called non-degenerate case when studying $U-$ statistics. In Theorem 1 it is a necessary condition, while assuming $E|h(X_1, X_2)|^{4/3} < \infty$ is close to being necessary, on account of the central limit theorem for U-statistics (see Borovskikh (2002), for example). Theorem 2 puts a totally new countenance on the classical theory of weak convergence for standardized U-type process as in Theorem 1 [see also Theorem A, Section 2.2.4 of Csörgő and Horváth (1997), Gombay and Horváth (1995, 2002)] in that here we derive results assuming only $g(X_1)$ being in the domain of attraction of the normal law.

This paper is organized as follows. In the next section we provide the proofs of main results. Then, in Section 3, we investigate the asymptotic behaviour of the $U-$type process $U_n(.)$ when it is based on kernels that are antisymmetric, i.e., $h(x,y)$ in such that $h(x,y) = -h(y,x)$. Throughout the paper $A, A_1, \ldots$ will be used to denote the constants which may be different at its each appearance.

2 Proofs of main results

We need some preliminaries to proving our main theorems. The following lemma, which is of independent interest, constitutes the key step.

Lemma 1 *Let $\psi(x,y)$ be a measurable real valued function for which we have*

$$E[\psi(X_1,X_2) \mid X_1] = E[\psi(X_1,X_2) \mid X_2] = 0$$

and $E|\psi(X_1,X_2)|^{4/3} < \infty$. Then, as $n \to \infty$,

$$\frac{1}{n} \max_{1\le k\le n-1} k^{-1/2} \Big| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i,X_j) \Big| = O_P(1), \tag{10}$$

$$\frac{1}{n^{3/2}} \max_{1\le k\le n-1} \Big| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i,X_j) \Big| = o_P(1). \tag{11}$$

Proof. We first prove (10). Let $\psi^*(X_i,X_j) = \psi(X_i,X_j) I_{|\psi|\le i^{3/2}} - E[\psi(X_i,X_j) I_{|\psi|\le i^{3/2}}]$, $g^*(X_i) = E(\psi^*(X_i,X_j)|X_i)$ and $\psi^{**}(X_i,X_j) = \psi^*(X_i,X_j) - g^*(X_i) - g^*(X_j)$. We have

$$\frac{1}{n} \max_{1\le k\le n-1} k^{-1/2} \Big| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i,X_j) \Big| \le I_1(n) + I_2(n) + I_3(n), \tag{12}$$

where

$$\begin{aligned}
I_1(n) &= \frac{1}{n} \max_{1\le k\le n-1} k^{-1/2} \Big| \sum_{i=1}^{k} \sum_{\substack{j\ne i\\ j=1}}^{k} \psi^{**}(X_i,X_j) \Big|, \\
I_2(n) &= \frac{1}{n} \max_{1\le k\le n-1} k^{-1/2} \Big| \sum_{i=1}^{k} \sum_{\substack{j=1\\ j\ne i}}^{n} \psi^{**}(X_i,X_j) \Big|, \\
I_3(n) &= \frac{1}{n} \max_{1\le k\le n-1} k^{-1/2} \Big| \sum_{i=1}^{k} \sum_{j=k+1}^{n} (\psi(X_i,X_j) - \psi^*(X_i,X_j) + g^*(X_i) + g^*(X_j)) \Big|.
\end{aligned}$$

We next prove $I_t(n) = O_P(1)$ for $t = 1,2,3$ and then (10) follows accordingly.

First consider $t = 1$. Write $Y_i = \sum_{j=1}^{i-1} \psi^{**}(X_i, X_j)$. It is readily seen that

$$\begin{aligned} E\Big|\sum_{i=2}^{\infty} i^{-3/2}\, Y_i\Big|^2 &= \sum_{i=2}^{\infty} i^{-3}\, EY_i^2 \le A \sum_{i=2}^{\infty} i^{-2}\, E[\psi^2(X_1, X_2) I_{|\psi| \le i^{3/2}}] \\ &\le A \sum_{k=1}^{\infty} E[\psi^2(X_1, X_2) I_{(k-1)^{3/2} < |\psi| \le k^{3/2}}] \sum_{i=k}^{\infty} i^{-2} \\ &\le A\, E|\psi(X_1, X_2)|^{4/3} < \infty. \end{aligned}$$

This, together with the Kronecker lemma, implies that $k^{-3/2} \sum_{i=1}^{k} Y_i \to 0$, a.s., and hence $I_1(n) = O_P(1)$, since $I_1(n) \le 2 \max_{1\le k\le n-1} k^{-3/2} |\sum_{i=2}^{k} Y_i|$.

Secondly we prove $I_2(n) = O_P(1)$. By noting that, for any a_i and $k \ge 1$, $\frac{1}{k}\sum_{i=1}^{k} a_i = b_k - \frac{1}{k}\sum_{i=1}^{k-1} b_i$, where $b_i = \sum_{t=1}^{i} a_t/t$, it follows that

$$I_2(n) \le \frac{1}{n^{1/2}} \max_{1\le k\le n-1} k^{-1} \Big|\sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\ne i}}^{n} \psi^{**}(X_i, X_j)\Big| \le \frac{2}{n^{1/2}} \max_{1\le k\le n-1} \Big|\sum_{i=1}^{k} \frac{1}{i} \sum_{\substack{j=1 \\ j\ne i}}^{n} \psi^{**}(X_i, X_j)\Big|.$$

Therefore, it only needs to be shown that, uniformly in $n \ge 1$,

$$\frac{1}{n^{1/2}} E\Big|\sum_{i=1}^{\infty} \frac{1}{i} \sum_{\substack{j=1 \\ j\ne i}}^{n} \psi^{**}(X_i, X_j)\Big| \le A < \infty.$$

This follows from a similar argument to that used in the proof for $I_2(n) = O_P(1)$. Indeed, for all $n \ge 1$, we have

$$\begin{aligned} \frac{1}{n^{1/2}} E\Big|\sum_{i=1}^{\infty} \frac{1}{i} \sum_{\substack{j=1 \\ j\ne i}}^{n} \psi^{**}(X_i, X_j)\Big| &\le \frac{1}{n^{1/2}} \Big[\sum_{i=1}^{\infty} \frac{1}{i^2} E\Big(\sum_{\substack{j=1 \\ j\ne i}}^{n} \psi^{**}(X_i, X_j)\Big)^2\Big]^{1/2} \\ &\le A\Big(\sum_{i=1}^{\infty} \frac{1}{i^2} E\psi^2(X_1, X_2) I_{(|\psi| \le i^{3/2})}\Big)^{1/2} \\ &< A(E|\psi(X_1, X_2)|^{4/3})^{1/2} < \infty. \end{aligned}$$

Finally we prove $I_3(n) = O_P(1)$. Noting $g^*(X_i) = E(\psi(X_i, X_j) I_{(|\psi| > i^{3/2})} | X_i)$ and recalling $E\psi(X_1, X_2) = 0$, it is readily seen that

$$\begin{aligned} I_3(n) &\le \frac{1}{n} \max_{1\le k\le n-1} k^{-1/2} \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j\ne i}}^{n} \Big[|\psi(X_i, X_j)| I_{|\psi| \ge i^{3/2}} + E(|\psi(X_i, X_j)| I_{(|\psi| > i^{3/2})} | X_i) \\ &\qquad + E(|\psi(X_i, X_j)| I_{(|\psi| > i^{3/2})} | X_j)\Big] + n^{-1/2} \sum_{i=1}^{n} E[|\psi(X_1, X_2)| I_{(|\psi| > i^{3/2})}] \\ &:= I_3^{(1)}(n) + I_3^{(2)}(n). \end{aligned} \tag{13}$$

By using similar arguments as in the proof of $I_1(n) = O_P(1)$ and the Kronecker lemma again, the claim that $I_3^{(1)}(n) = O_P(1)$ follows from

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{\infty} \frac{1}{i^{1/2}} \sum_{\substack{j=1\\ j\neq i}}^{n} E[|\psi(X_1,X_2)| I_{(|\psi|>i^{3/2})}] \ &\leq\ \sum_{i=1}^{\infty} \frac{1}{i^{1/2}} E[|\psi(X_1,X_2)| I_{(|\psi|>i^{3/2})}] \\
&\leq\ \sum_{k=1}^{\infty} E[|\psi(X_1,X_2)| I_{(k^{3/2}<|\psi|\leq (k+1)^{3/2})}] \sum_{i=1}^{k} \frac{1}{i^{1/2}} \\
&\leq\ A\, E|\psi(X_1,X_2)|^{4/3} < \infty,
\end{aligned}
$$

uniformly for all $n \geq 1$. As to $I_3^{(2)}(n)$, it is obvious that

$$
I_3^{(2)}(n) \leq n^{-1/2} \sum_{i=1}^{n} i^{-1/2}\, E|\psi(X_1,X_2)|^{4/3} \leq A\, E|\psi(X_1,X_2)|^{4/3} = O(1).
$$

Taking these estimates into (13), we obtain $I_3(n) = O_P(1)$. The proof of (10) is now complete.

The proof of (11) is similar to that of (10), but we have to use a different truncation. In the following, we let $\psi^*(X_i,X_j) = \psi(X_i,X_j) I_{|\psi|\leq n^{3/2}} - E[\psi(X_i,X_j) I_{|\psi|\leq n^{3/2}}]$, $g^*(X_i) = E(\psi^*(X_i,X_j)|X_i)$ and $\psi^{**}(X_i,X_j) = \psi^*(X_i,X_j) - g^*(X_i) - g^*(X_j)$. It follows easily that

$$
\frac{1}{n^{3/2}} \max_{1\leq k\leq n-1} \Big| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i,X_j) \Big| \ \leq\ \frac{1}{2}\Big[I_0^*(n) + I_1^*(n) + I_2^*(n) \Big] + I_3^*(n), \qquad (14)
$$

where $I_0^*(n) = \frac{1}{n^{3/2}} \Big| \sum_{i=1}^{n} \sum_{\substack{j\neq i\\ j=1}}^{n} \psi^{**}(X_i,X_j) \Big|$,

$$
\begin{aligned}
I_1^*(n) &= \frac{1}{n^{3/2}} \max_{1\leq k\leq n-1} \Big| \sum_{i=1}^{k} \sum_{\substack{j\neq i\\ j=1}}^{k} \psi^{**}(X_i,X_j) \Big|, \\
I_2^*(n) &= \frac{1}{n^{3/2}} \max_{1\leq k\leq n-1} \Big| \sum_{i=k+1}^{n} \sum_{\substack{j=k+1\\ j\neq i}}^{n} \psi^{**}(X_i,X_j) \Big|, \\
I_3^*(n) &= \frac{1}{n^{3/2}} \max_{1\leq k\leq n-1} \Big| \sum_{i=1}^{k} \sum_{j=k+1}^{n} (\psi(X_i,X_j) - \psi^*(X_i,X_j) + g^*(X_i) + g^*(X_j)) \Big|.
\end{aligned}
$$

It is readily seen that

$$
\begin{aligned}
E[I_0^*(n)]^2 &\leq A\, n^{-1}\, E\psi^2(X_1,X_2) I_{|\psi|\leq n^{3/2}} \\
&\leq A\Big[\epsilon^{-2}\, n^{-1/3}\, E|\psi(X_1,X_2)|^{4/3} + E|\psi(X_1,X_2)|^{4/3} I_{|\psi|\geq n}\Big] \\
&\to 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

This yields that $I_0^*(n) = o_P(1)$. Noting that $\{\sum_{j=2}^k Y_j, \mathcal{F}_k, 2 \le k \le n\}$ is a martingale, where $Y_j = \sum_{i=1}^{j-1} \psi^{**}(X_i, X_j)$ and $\mathcal{F}_k = \sigma\{X_1, ..., X_k\}$, it follows from the well-known Maximum inequality for martingales that, for any $\epsilon > 0$,

$$\begin{aligned} P(I_1^*(n) \ge \epsilon) &\le 4\epsilon^{-2} n^{-3} E \max_{1\le k\le n-1} |\sum_{j=2}^k Y_j|^2 \le A\,\epsilon^{-2} n^{-3} \sum_{j=2}^n EY_j^2 \\ &\le A\,\epsilon^{-2} n^{-1} E\psi^2(X_1, X_2) I_{|\psi|\le n^{3/2}} \\ &\le A\left[\epsilon^{-2} n^{-1/3} E|\psi(X_1, X_2)|^{4/3} + E|\psi(X_1, X_2)|^{4/3} I_{|\psi|\ge n}\right] \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

This yields $I_1^*(n) = o_P(1)$. By a similar argument as in the proof for $I_1^*(n) = o_P(1)$, we have $I_2^*(n) = o_P(1)$. As for $I_3^*(n)$, it is readily seen that

$$\begin{aligned} E|I_3^*(n)| &\le \frac{1}{n^{3/2}} \sum_{i=1}^n \sum_{\substack{j=1\\ j\ne i}}^n E\left|\psi(X_i, X_j) - \psi^*(X_i, X_j) + g^*(X_i) + g^*(X_j)\right| \\ &\le 4\,n^{1/2} E[|\psi(X_1, X_2)| I_{|\psi|\ge n^{3/2}}] \\ &\le 4\,E[|\psi(X_1, X_2)|^{4/3} I_{|\psi|\ge n^{3/2}}] \to 0, \end{aligned}$$

as $n \to \infty$, which implies that $I_3^*(n) = o_P(1)$. Taking all estimates for $I_t^*(n), t = 0, 1, 2, 3$ in to (14), we obtain the required (11). The proof of Lemma 1 is now complete.

The next two lemmas are due to CsCsHM (1986) [cf. Lemma A.5.1 and Theorem A.5.1 respectively in Csögő and Horváth (1997)]. Proofs of Lemmas 2 and 3 can also be found in Section 4.1 of Csörgő and Horváth (1993).

Lemma 2 *Let $q(t) \in Q$. If $I(q, c) < \infty$ for some $c > 0$, then*

$$\lim_{t\downarrow 0} t^{1/2}/q(t) = 0 \quad \textit{and} \quad \lim_{t\uparrow 1} (1-t)^{1/2}/q(t) = 0.$$

Lemma 3 *Let $\{W(t), 0 \le t < \infty\}$ be a standard Wiener process and $q(t) \in Q$. Then,*

(a) $I(q, c) < \infty$ for any $c > 0$ if and only if

$$\limsup_{t\downarrow 0} |W(t)|/q(t) = 0, \ a.s. \quad \textit{and} \quad \limsup_{t\uparrow 1} |W(1) - W(t)|/q(t) = 0, \ a.s.$$

(b) $I(q, c) < \infty$ for some $c > 0$ if and only if

$$\limsup_{t\downarrow 0} |W(t)|/q(t) < \infty, \ a.s. \quad \textit{and} \quad \limsup_{t\uparrow 1} |W(1) - W(t)|/q(t) < \infty, \ a.s.$$

We are now ready to prove our main theorems.

Proof of Theorem 1. Together with the notation as in Section 1, we write $\psi(x,y) = h(x,y) - \theta - g(x) - g(y)$ and $T_n(t) = W_{[(n+1)t]}, 0 \le t \le 1$, where

$$W_k = (n-k)\sum_{j=1}^{k} g(X_j) + k\sum_{j=k+1}^{n} g(X_j).$$

Noting that $g(X_j)$ are iid random variables with $Eg(X_1) = 0$ and $\sigma^2 = Eg^2(X_1) < \infty$, as in the proof of (2.1.45) in Csörgő and Horváth (1997), on an appropriate probability space for $X, X_1, X_2, \cdots$ we can define a sequence of Gaussian processes $\{\Gamma_n(t), 0 \le t \le 1\}$ such that, for each $n \ge 1$,

$$\{\Gamma_n(t), 0 \le t \le 1\} \stackrel{D}{=} \{\Gamma(t), 0 \le t \le 1\},$$

and if $q \in Q$ and $I(q,c) < \infty$ for some $c > 0$, then, as $n \to \infty$,

$$\sup_{1/n \le t \le (n-1)/n} \Big| n^{-3/2}\sigma^{-1}T_n(t) - \Gamma_n(t) \Big| \Big/ q(t) \;=\; o_P(1). \tag{15}$$

By virtue of (15), Theorem 1 will follow if we prove

$$J_n \;:=\; \sup_{1/n \le t \le (n-1)/n} \Big| n^{-3/2}U_n(t) - n^{-3/2}T_n(t) \Big| \Big/ q(t) \;=\; o_P(1). \tag{16}$$

In order to prove (16), write $V_n(t) = W^*_{[(n+1)t]}$, where $W_k^* = \sum_{j=1}^{k}\sum_{j=k+1}^{n}\psi(X_i, X_j)$. Note that $E(\psi(X_1,X_2) \mid X_1) = E(\psi(X_1,X_2) \mid X_2) = 0$ and

$$E|\psi(X_1,X_2)|^{4/3} \;\le\; A\, E|h(X_1,X_2)|^{4/3} \;<\; \infty.$$

It follows from (11) that

$$\begin{aligned} J_n^{(1)} \;&:=\; \sup_{\delta \le t \le 1-\delta} |n^{-3/2}V_n(t)| \Big/ q(t) \\ &\le\; \frac{1}{n^{3/2}} \max_{1 \le k \le n-1} \Big| \sum_{i=1}^{k}\sum_{j=k+1}^{n}\psi(X_i,X_j) \Big| \sup_{\delta \le t \le 1-\delta} q^{-1}(t) = o_P(1), \end{aligned}$$

for any $\delta \in (0,1)$ and $q \in Q$. Let $\delta > 0$ be so small that $q(t)$ is already nondecreasing on $(0,\delta)$ and nonincreasing on $(1-\delta, 1)$ and let n be so large such that $1/n \le \delta$. It follows from (10) and Lemma 2 that

$$\begin{aligned} J_n^{(2)} \;&:=\; \sup_{0 < t \le \delta} |n^{-3/2}V_n(t)| \Big/ q(t) \\ &\le\; \frac{1}{n} \max_{1 \le k \le n-1} k^{-1/2} \Big| \sum_{i=1}^{k}\sum_{j=k+1}^{n}\psi(X_i,X_j) \Big| \sup_{0 < t \le \delta} t^{1/2}/q(t) = o_P(1), \end{aligned}$$

when $n \to \infty$ and then $\delta \to 0$. Similarly, we have also

$$\begin{aligned} J_n^{(3)} &:= \sup_{1-\delta \le t < 1} |n^{-3/2} V_n(t)| \Big/ q(t) \\ &\le \frac{1}{n} \max_{1 \le k \le n-1} k^{-1/2} \Big| \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) \Big| \sup_{1-\delta \le t < 1} t^{1/2}/q(t) = o_P(1), \end{aligned}$$

when $n \to \infty$ and then $\delta \to 0$. By virtue of these estimates, it is readily seen that

$$J_n \le J_n^{(1)} + J_n^{(2)} + J_n^{(3)} + A\, n^{-1/2} \sup_{1/n \le t \le (n-1)/n} 1/q(t) = o_P(1), \tag{17}$$

which yields (16). The proof of Theorem 1 is now complete.

Proof of Corollary 1. Having Theorem 1, Lemmas 2-3 and the result (16), the proof of Corollary 1 is the same as that given in the proof of Theorem 2.4.2 in Csörgő and Horváth (1997), and hence the details are omitted.

Proof of Theorem 2. We first prove (7). It is readily seen that

$$\begin{aligned} \hat{U_n}(t) &= n^{-3/2} (\hat\sigma)^{-1} \{ Z_{[(n+1)t]} - n^2 t(1-t)\theta \} + t(1-t) n^{1/2} (\hat\sigma)^{-1} (\hat\theta - \theta) \\ &= \left\{ \frac{\sum_{j=1}^n g^2(X_j)}{n\hat\sigma^2} \right\}^{1/2} n^{-1} \Big\{ \sum_{j=1}^n g^2(X_j) \Big\}^{-1/2} U_n(t) + t(1-t) n^{1/2} (\hat\sigma)^{-1} (\hat\theta - \theta). \end{aligned} \tag{18}$$

Furthermore $U_n(t) = T_n(t) + V_n(t)$, where $T_n(t)$ and $V_n(t)$ are defined as in the proof of Theorem 1. Recalling that $g(X_1)$ is in the domain of attraction of the normal law, as in the proof of Theorem 5.2 of Csörgő, Szyszkowicz and Wang [CsSzW] (2004) with minor modifications, we have that on an appropriate probability space for $X, X_1, X_2, \cdots$, we can define a sequence of Gaussian processes $\{\Gamma_n(t), 0 \le t \le 1\}$ such that (2) holds true, and as $n \to \infty$,

$$\sup_{0<t<1} \Big| n^{-1} \Big\{ \sum_{j=1}^n g^2(X_j) \Big\}^{-1/2} T_n(t) - \Gamma_n(t) \Big| \Big/ q(t) = o_P(1),$$

if and only if $I(q, c) < \infty$ for any $c > 0$. Therefore, to prove (7), it suffices to show that

$$n^{-1} \Big\{ \sum_{j=1}^n g^2(X_j) \Big\}^{-1/2} \sup_{0<t<1} |V_n(t)|/q(t) = o_P(1), \tag{19}$$

$$\Big\{ n^{-1} \sum_{j=1}^n g^2(X_j) \Big\}^{-1} \hat\sigma^2 - 1 = o_P(1), \tag{20}$$

and

$$n^{1/2}(\hat{\sigma})^{-1}(\hat{\theta}-\theta) \;=\; o_P(1). \tag{21}$$

The proof of (19) is simple and in fact (19) holds true if $q(x)$ satisfies $I(q,c)<\infty$ for some $c>0$. Indeed, since $g(X_1)$ is in the domain of attraction of the normal law, we have $\frac{1}{b_n}\sum_{j=1}^n g^2(X_j) \to_P 1$, where $b_n = n\,l(n)$ with that $l(n)=Eg^2(X_1)$ if $Eg^2(X_1)<\infty$ or $l(n)\to\infty$ if $Eg^2(X_1)=\infty$. On the other hand, as in the proof of (16), $n^{-3/2}\sup_{0<t<1}|V_n(t)|/q(t) = o_P(1)$ even when $q(x)$ satisfies $I(q,c)<\infty$ for some $c>0$, and hence (19) follows immediately from these facts.

We next prove (20). The claim (21) follows by using (20), and hence the details are omitted. Without loss of generality, we assume $\theta=0$. We may rewrite $\hat{\sigma}^2$ as

$$\begin{aligned}
\hat{\sigma}^2 &= \frac{1}{n(n-1)^2}\sum_{i\neq j\neq k} h(X_i,X_j)h(X_i,X_k) + \frac{1}{n(n-1)^2}\sum_{i\neq j} h^2(X_i,X_j) - \hat{\theta}^2 \\
&:= W_{n1}+W_{n2}-\hat{\theta}^2.
\end{aligned}$$

Recalling $E|h(X_1,X_2)|^{5/3}<\infty$, it follows from a Marcinkiewicz type strong law for U-statistics that $W_{n2}-\hat{\theta}^2\to 0, a.s.$ [see Gine and Zinn (1992), for example]. Therefore (20) will follow if we prove

$$\Big\{n^{-1}\sum_{j=1}^n g^2(X_j)\Big\}^{-1} W_{n1}-1 \;=\; o_P(1). \tag{22}$$

Write, for $i\neq j\neq k$,

$$\begin{aligned}
h_{ij}^{(1)} &= h(X_i,X_j)I_{(|h|\leq n^{6/5})}, \qquad g^{(1)}(X_i) \;=\; E(h_{ij}^{(1)}|X_i), \\
\psi_{ijk} &= h_{ij}^{(1)}\,h_{ik}^{(1)} - Eh_{ij}^{(1)}\,h_{ik}^{(1)}, \\
\varphi_i^{(1)} &= E(\psi_{ijk}|X_i), \quad \varphi_j^{(2)} \;=\; E(\psi_{ijk}|X_j), \quad \varphi_k^{(3)} \;=\; E(\psi_{ijk}|X_k).
\end{aligned}$$

Noting that $E\{h_{ij}^{(1)}\,h_{ik}^{(1)}|X_i\} = \{g^{(1)}(X_i)\}^2$, it is readily seen that $\varphi_i^{(1)} = \{g^{(1)}(X_i)\}^2 - E[h_{ij}^{(1)}\,h_{ik}^{(1)}]$, and

$$\begin{aligned}
\sum_{i\neq j\neq k} h_{ij}^{(1)}\,h_{ik}^{(1)} &= \sum_{i\neq j\neq k}\psi_{ijk} + \sum_{i\neq j\neq k} E[h_{ij}^{(1)}\,h_{ik}^{(1)}] \\
&= \sum_{i\neq j\neq k}\{g^{(1)}(X_i)\}^2 + \sum_{i\neq j\neq k}\{\varphi_j^{(2)}+\varphi_k^{(3)}\} \\
&\quad + \sum_{i\neq j\neq k}(\psi_{ijk}-\varphi_i^{(1)}-\varphi_j^{(2)}-\varphi_k^{(3)}) \\
&:= V_{n1}+V_{n2}+V_{n3}.
\end{aligned}$$

In the next paragraph, we will show that

$$\Big\{n^{-1}\sum_{j=1}^{n} g^2(X_j)\Big\}^{-1}\Big(n^{-3}\,V_{n1}\Big)-1 \;=\; o_P(1), \tag{23}$$

$$n^{-3}\,(V_{n2}+V_{n3}) \;=\; o_P(1). \tag{24}$$

It follows from (23) and (24) that

$$\Big\{n^{-1}\sum_{j=1}^{n} g^2(X_j)\Big\}^{-1} n^{-3}\sum_{i\neq j\neq k} h_{ij}^{(1)}\,h_{ik}^{(1)} \;-\; 1 \;=\; o_P(1), \tag{25}$$

and then (22) follows from (25) and

$$\begin{aligned} P\left(\sum_{i\neq j\neq k} h_{ij}\,h_{ik} \neq \sum_{i\neq j\neq k} h_{ij}^{(1)}\,h_{ik}^{(1)}\right) &\le n^2\,P(|h(X_1,X_2)|\ge n^{6/5}) \\ &\le E|h(X_1,X_2)|^{5/3} I_{|h|\ge n^{6/5}} \to 0. \end{aligned}$$

We are to prove (23) and (24) now. Consider (23) first. By noting that $g^{(1)}(X_1) = g(X_1)-g^*(X_j)$, where $g^*(X_j)=E\{h(X_1,X_2)I_{(|h|\ge n^{6/5})}|X_1\}$, we have

$$\begin{aligned} \Big|\sum_{j=1}^{n}\Big[\{g^{(1)}(X_j)\}^2-g^2(X_j)\Big]\Big| &\le \sum_{j=1}^{n}\Big[2|g(X_j)|\,|g^*(X_j)|+|g^*(X_j)|^2\Big] \\ &\le 2\Big[\sum_{j=1}^{n} g^2(X_j)\Big]^{1/2}\Big[\sum_{j=1}^{n}\{g^*(X_j)\}^2\Big]^{1/2}+\sum_{j=1}^{n}\{g^*(X_j)\}^2. \end{aligned}$$

Now, since $g(X_1)$ is in the domain of attraction of the normal law [which implies that $\frac{1}{n}\sum_{j=1}^{n} g^2(X_j)\to_P C>0$, where C may be ∞], simple calculations show that (23) will follow if we prove

$$\frac{1}{n}\sum_{j=1}^{n}\{g^*(X_j)\}^2 = o_P(1). \tag{26}$$

In fact, for any $\epsilon>0$, we have

$$\begin{aligned} P\Big(\sum_{j=1}^{n}\{g^*(X_j)\}^2\ge \epsilon\,n\Big) &\le \epsilon^{-1/2}n^{-1/2}\sum_{j=1}^{n} E|g^*(X_j)| \\ &\le \epsilon^{-1/2}n^{1/2}E|h(X_1,X_2)|I_{(|h|\ge n^{6/5})} \\ &\le \epsilon^{-1/2}\,E|h(X_1,X_2)|^{5/3}I_{(|h|\ge n^{6/5})}\to 0, \end{aligned}$$

as $n\to\infty$. This implies (26) and hence completes the proof of (23).

We next prove (24). By noting that $n^{-3}V_{n3}$ is a degenerate U-statistic of order 3, it follows from moment inequality for degenerate U-statistics (see, Borovskikh (1996), for example) that, for any $\epsilon > 0$,

$$
\begin{aligned}
P(|V_{n3}| \geq \epsilon n^3) &\leq \epsilon^{-5/3}\, n^{-5}\, E|V_{n3}|^{5/3} \\
&\leq A\, \epsilon^{-5/3}\, n^{-2}\, E\Big|\psi_{123} - \varphi_1^{(1)} - \varphi_2^{(2)} - \varphi_3^{(3)}\Big|^{5/3} \\
&\leq A\, \epsilon^{-5/3}\, n^{-2}\, E|h(X_1, X_2)|^{10/3} I_{(|h| \leq n^{6/5})} \\
&\leq A\, \epsilon^{-5/3} \Big[n^{-1/3} + E|h(X_1, X_2)|^{5/3} I_{(|h| \geq n^{1/2})}\Big] \to 0, \qquad (27)
\end{aligned}
$$

as $n \to \infty$. On the other hand, by noting that

$$
\begin{aligned}
E\Big\{E\Big[h_{12}^{(1)}\, h_{13}^{(1)} | X_2\Big]\Big\}^2 &= E\Big\{h_{12}^{(1)}\, h_{13}^{(1)}\, E\Big[h_{42}^{(1)}\, h_{45}^{(1)} | X_2\Big]\Big\} \\
&= E\Big[h_{12}^{(1)}\, h_{13}^{(1)}\, h_{42}^{(1)}\, h_{45}^{(1)}\Big] \\
&\leq \Big[E h^2(X_1, X_2) I_{|h| \leq n^{6/5}}\Big]^2 \leq n^{4/5} \left\{E|h(X_1, X_2)|^{5/3}\right\}^2,
\end{aligned}
$$

it is readily seen that, for any $\epsilon > 0$,

$$
\begin{aligned}
P(|V_{n3}| \geq \epsilon n^3) &\leq \epsilon^{-2}\, E\Big(n^{-3} V_{n2}\Big)^2 \\
&\leq A\, \epsilon^{-2}\, n^{-1} E\Big(\varphi_1^{(2)} + \varphi_1^{(3)}\Big)^2 \\
&\leq A\, \epsilon^{-2}\, n^{-1} \left[E\Big\{E\Big(h_{12}^{(1)}\, h_{13}^{(1)} | X_2\Big)\Big\}^2 + \Big(E\Big\{h_{12}^{(1)}\Big\}^2\Big)^2\right] \\
&\leq A\, \epsilon^{-2}\, n^{-1/5} \left\{E|h(X_1, X_2)|^{5/3}\right\}^2 \to 0, \qquad (28)
\end{aligned}
$$

as $n \to \infty$. By virtue of (27) and (28), we obtain (24). The proof of (7) is now complete.

The result (8) is a direct consequence of (7). As for (9), by virtue of (18)-(21) (recalling that (19) still holds true for $q(x)$ satisfying $I(q, c) < \infty$ for some $c > 0$, as explained in its proof), it suffices to show that

$$
\sup_{0<t<1} \Big| n^{-1} \Big\{ \sum_{j=1}^{n} g^2(X_j) \Big\}^{-1/2} T_n(t) \Big| \;\; \to_D \;\; \sup_{0<t<1} |\Gamma(t)|/q(t) \qquad (29)
$$

if and only if $I(q, c) < \infty$ for some $c > 0$, where $T_n(t) = W_{[(n+1)t]}, 0 \leq t \leq 1$, with

$$
W_k = (n-k) \sum_{j=1}^{k} g(X_j) + k \sum_{j=k+1}^{n} g(X_j).
$$

This follows from the same arguments as in the proof of Corollary 5.2 in CsSzW (2004), and hence the details are omitted. This also completes the proof of Theorem 2.

3 Antisymmetric kernel

In this section we consider the asymptotics of U-type processes with antisymmetric kernel $h(x,y)$, i.e., $h(x,y) = -h(y,x)$. This kind of kernels can not be symmetrized, but they are especially useful to check the equality of distributions for different groups of random variables since $Eh(X_1, X_2) = 0$ whenever $X_1 =_D X_2$. An example is given in Pettitt (1979), who used functions of the Mann-Whitney type statistics

$$R_n(t) \quad = \quad (12)^{1/2} n^{-3/2} \sum_{1 \le i \le nt} \sum_{nt < j \le n} \operatorname{sign}(X_i - X_j)$$

to detect possible changes in distribution.

For the anti-symmetric kernel $h(x,y)$, by letting $g(t) = Eh(X_1, t)$, we may write

$$Z_k = \sum_{i=1}^{k} \sum_{j=k+1}^{n} \psi(X_i, X_j) + n \Big[\sum_{i=1}^{k} g(X_i) - \frac{k}{n} \sum_{i=1}^{n} g(X_i) \Big],$$

where $\psi(x,y) = h(x,y) - g(x) + g(y)$ with

$$E\left[\psi(X_1, X_2) \mid X_1\right] = E\left[\psi(X_1, X_2) \mid X_2\right] = 0.$$

Since Lemma 1 does not depend on the symmetry of the kernel, similarly to the proofs of Theorems 1 and 2, we have the following results for U-type processes with antisymmetric kernel $h(x,y)$, which improve and generalize the similar earlier results of Csörgő and Horváth (1988a, b), Szyszkowicz (1991, 1992) and those given in Section 2.4 of Csörgő and Horváth (1997) along these lines. It is interesting to note that the Gaussian limit process that is shared by Theorems 1 and 2 and that of Theorems 3 and 4 are different, although they are of equal variance. For further related results, we refer to Janson and Wichura (1983), and Gombay (2000a, b, 2001, 2004).

We continue to use the notations as in Section 1, but $U_n(t)$ and $\hat{U}_n(t)$ are now defined in terms of antisymmetric kernel $h(x,y) = -h(y,x)$.

Theorem 3 *Let $q \in Q$. Assume H_0, $0 < \sigma^2 < \infty$ and $E|h(X_1, X_2)|^{4/3} < \infty$. Then, on an appropriate probability space for $X, X_1, X_2, \cdots$, we can define a sequence of Brownian bridges $\{B_n(t), 0 \le t \le 1\}$ such that if $I(q,c) < \infty$ for some $c > 0$, then as $n \to \infty$,*

$$\sup_{1/n \le t \le (n-1)/n} \Big| n^{-3/2} \sigma^{-1} U_n(t) - B_n(t) \Big| \Big/ q(t) \quad = \quad o_P(1). \tag{30}$$

Consequently, we have that

(a) a sequence of Brownian bridges $\{B_n(t), 0 \le t \le 1\}$ *can be defined such that (30) holds true if and only if* $I(q,c) < \infty$ *for any* $c > 0$*;*

(b) *as* $n \to \infty$*,*

$$n^{-3/2}\sigma^{-1}U_n(t)\Big/q(t) \quad \Rightarrow \quad B(t)\Big/q(t) \tag{31}$$

on $(D[0,1], \rho)$*, where* ρ *is the sup-norm metric for functions in* $D[0,1]$*, if and only if* $I(q,c) < \infty$ *for any* $c > 0$*;*

(c) as $n \to \infty$*,*

$$n^{-3/2}\sigma^{-1}\sup_{0<t<1}|U_n(t)|\Big/q(t) \quad \to_D \quad \sup_{0<t<1}|B(t)|/q(t) \tag{32}$$

if and only if $I(q,c) < \infty$ *for some* $c > 0$*, where, in (b) and (c),* $\{B(t), 0 \le t \le 1\}$ *is a Brownian bridge.*

Theorem 3 is to be compared to Szyszkowicz (1991, Theorem 2.1) [cf. Theorem 2.4.1 in Csörgő and Horváth (1997)].

Theorem 4 *Let* $q \in Q$*. Assume* H_0*,* $E|h(X_1,X_2)|^{5/3} < \infty$ *and that* $g(X_1)$ *is in the domain of attraction of the normal law. Then, on an appropriate probability space for* $X, X_1, X_2, \cdots$*, we can define a sequence of Brownian bridges* $\{B_n(t), 0 \le t \le 1\}$ *such that, as* $n \to \infty$*,*

$$\sup_{0<t<1}\Big|\hat{U}_n(t) - B_n(t)\Big|\Big/q(t) \quad = \quad o_P(1), \tag{33}$$

if and only if $I(q,c) < \infty$ *for any* $c > 0$*. Consequently, as* $n \to \infty$*,*

$$\hat{U}_n(t)/q(t) \quad \Rightarrow \quad B(t)\Big/q(t), \quad \text{on } (D[0,1], \rho) \tag{34}$$

if and only if $I(q,c) < \infty$ *for any* $c > 0$*, where* $\{B(t), 0 \le t \le 1\}$ *is a Brownian bridge. Furthermore we also have*

$$\sup_{0<t<1}|\hat{U}_n(t)|\Big/q(t) \quad \to_D \quad \sup_{0<t<1}|B(t)|/q(t) \tag{35}$$

if and only if $I(q,c) < \infty$ *for some* $c > 0$*.*

On taking $h(x,y) = x - y$, Theorem 4 essentially extends Corollary 2.1.1 of Csörgő and Horváth (1997) [cf. Theorem 5.1 in CsSzW (2004)] and rhymes with Theorem 5.2 of CsSzW (2004) [cf. also CsSzW (2006)], where we study directly the problem of change in the mean in the domain of attraction of the normal law.

REFERENCES

Borovskikh, Yu. V.(1996). *U-statistics in Banach spaces.* VSP, Utrecht.

Borovskikh, Yu. V. (2002). On the normal approximation of U-statistics. *Theory Probab. Appl.* **45**, 406–423.

Csörgő, M., Csörgő, S., Horváth, L. and Mason, D. (1986). Weighted empirical and quantile processes, *Ann. Probab.* **14**, 31-85.

Csörgő, M. and Horváth, L. (1988a). Invariance principles for changepoint problems. *J. Multivariate Anal.* **27**, 151–168.

Csörgő, M. and Horváth, L. (1988b). Nonparametric methods for changepoint problems, In *Handbook of Statistics*, Elsevier Science Publisher B.V., 403-425, North-Holland, Amsterdam.

Csörgő, M. and Horváth, L. (1993). *Weighted Approximations in Probability and Statistics*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, Wiley, Chichester.

Csörgő, M., and Horváth, L. (1997). *Limit Theorems in Change-Point Analysis*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, Wiley, Chichester.

Csörgő, M., Norvaiša, R. and Szyszkowicz, B. (1999). Convergence of weighted partial sums when the limiting distribution is not necessarily Radon, *Stochastic Process. Appl.* **81**, 81-101.

Csörgő, M., Szyszkowicz, B. and Wang, Q. (2004). On weighted approximations and strong limit theorems for self-normalized partial sums processes. In *Asymptotic methods in stochastics,* 489–521, Fields Inst. Commun. **44**, Amer. Math. Soc., Providence, RI.

Csörgő, M., Szyszkowicz, B. and Wang, Q. (2006). Change in the mean in the domain of attraction of the normal law. *Austrian Journal of Statistics*, **35**, 93-103.

Gine, E., and Zinn, J. (1992). Marcinkiewicz type laws of large numbers and convergence of moments for U-statistics. In *Probability in Banach Spaces* (R. Dudley, M. Hahn and J. Kuelbs, eds) **8** 273-291, Birkhauser, Boston.

Gombay, E. (2000a). Comparison of U-statistics in the change-point problem and in sequential change detection. Endre Cski 65. *Period. Math. Hungar.* **41**, 157–166.

Gombay, E. (2000b). U-statistics for sequential change detection. *Metrika* **52**, 133–145.

Gombay, E. (2001). U-statistics for change under alternatives. *J. Multivariate Anal.* **78**, 139–158.

Gombay, E. (2004). U-statistics in sequential tests and change detection. Abraham Wald centennial celebration: invited papers. Part II. *Sequential Anal.* **23**, 257–274.

Gombay, E. and Horváth, L. (1995). An application of U-statistics to change-point analysis. *Acta Sci. Math.* (Szeged) **60**, 345–357.

Gombay, E. and Horváth, L. (2002). Rates of convergence for U-statistic processes and their bootstrapped versions. *J. Statist. Plann. Inference* **102**, 247–272.

Janson, S. and Wichura, M. J. (1984). Invariance principles for stochastic area and related stochastic integrals. *Stochastic Process. Appl.* **16**, 71–84.

Pettitt, A. N. (1979). A nonparametric approach to the change-point problem. *J. Roy. Statist. Soc. Ser. C* **28**, 126–135.

Szyszkowicz, B. (1991). Weighted stochastic processes under contiguous alternatives, *C.R. Math. Rep. Acad. Sci. Canada* **13**, 211-216.

Szyszkowicz, B. (1992). *Weak Convergence of Stochastic Processes in Weighted Metrics and their Applications to Contiguous Changepoint Analysis.* Ph. D. Dissertation, Carleton University.

Szyszkowicz, B. (1996). Weighted approximations of partial sum processes in $D[0,\infty)$. I, *Studia Sci. Math. Hungar.* **31**, 323-353.

Szyszkowicz, B. (1997). Weighted approximations of partial sum processes in $D[0,\infty)$. II, *Studia Sci. Math. Hungar.* **33**, 305-320.

Miklós Csörgő and Barbara Szyszkowicz

School of Mathematics and Statistics
Carleton University
1125 Colonel By Drive
Ottawa, K1S 5B6
Canada

Qiying Wang

School of Mathematics and Statistics
The University of Sydney
NSW 2006, Australia