

Functional limit theorems for occupation time fluctuations of branching systems in the cases of large and critical dimensions

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Abstract

We give functional limit theorems for the fluctuation of the rescaled occupation time process of a critical branching particle system in \mathbb{R}^d with symmetric α -stable motion, in the cases of critical and large dimensions, $d = 2\alpha$ and $d > 2\alpha$. In a previous paper (Bojdecki et al, 2004b) we treated the case of intermediate dimensions, $\alpha < d < 2\alpha$, which leads to a long-range dependence limit process. In contrast, in the present cases the limits are generalized Wiener processes. We use the same space-time random field method of the previous paper, the main difference being that now the tightness requires a new approach and the proofs are more difficult. We also give analogous results for the system without branching in the cases $d = \alpha$ and $d > \alpha$.

MSC: primary 60F17, 60G20, secondary 60G15, 60G60

Keywords: Functional central limit theorem; Occupation time fluctuation; Branching particle system; Generalized Wiener process, critical dimension

1. Introduction

In Bojdecki et al (2004b) (denoted for brevity in the sequel by BGT), we proved a functional central limit theorem for the rescaled occupation time process of a critical binary branching particle system in \mathbb{R}^d with particles moving according to a symmetric α -stable Lévy process, in the case of intermediate dimensions, $\alpha < d < 2\alpha$, which leads to a long-range dependence limit process. In the present paper we consider the same problem in the cases of critical and large dimensions, $d = 2\alpha$ and $d > 2\alpha$, where long-range dependence no longer appears. There are significant differences in the types of the results and in some aspect of the proofs. Let us summarize the main differences between the limiting behaviors for different ranges of the parameters d and α .

The rescaled occupation time fluctuation process is defined by

$$X_T(t) = \frac{1}{F_T} \int_0^{Tt} (N_s - EN_s) ds, \quad t \geq 0, \quad (1.1)$$

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¹ Research partially supported by KBN grant 2P03A02622 (Poland).

² Research partially supported by CONACyT grant 37130-E (Mexico).

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where $(N_t)_{t \geq 0}$ is the empirical measure process of the system, $T > 0$ (T will tend to ∞), and F_T is a suitable norming. For the initial state N_0 we take a standard Poisson random field (i.e., with intensity the Lebesgue measure λ). Weak convergence (denoted by \Rightarrow) of the process X_T takes place in the space $C([0, \tau], \mathcal{S}'(\mathbb{R}^d))$ for any $\tau > 0$, where $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions, dual of the space $\mathcal{S}(\mathbb{R}^d)$ of smooth rapidly decreasing functions.

(1) $\alpha < d < 2\alpha$. With the norming $F_T = T^{(3-d/\alpha)/2}$, we have $X_T \Rightarrow C\lambda\zeta$ as $T \rightarrow \infty$, where C is a constant and $\zeta = (\zeta_t)_{t \geq 0}$ is a real long-range dependence centered Gaussian process called sub-fractional Brownian motion, whose covariance is given by

$$s^h + t^h - \frac{1}{2}[(s+t)^h + |s-t|^h], \quad s, t \geq 0,$$

with $h = 3 - d/\alpha$. The properties of the process ζ , in particular the long-range dependence, are studied in Bojdecki et al (2004a).

(2) $d = 2\alpha$. With the norming $F_T = (T \log T)^{1/2}$, we have $X_T \Rightarrow C\lambda\beta$ as $T \rightarrow \infty$, where C is a constant and $\beta = (\beta_t)_{t \geq 0}$ is real standard Brownian motion.

(3) $d > 2\alpha$. With the norming $F_T = T^{1/2}$, we have $X_T \Rightarrow X$ as $T \rightarrow \infty$, where X is a “truly” generalized Wiener process (i.e., $\mathcal{S}'(\mathbb{R}^d)$ -valued but not measure-valued).

Thus, for $\alpha < d < 2\alpha$ the spatial structure of the limit is simple (the measure λ) and the temporal structure is complicated (with long-range dependence). For $d = 2\alpha$ the spatial structure is simple (λ) and so is the temporal structure (with stationary independent increments). For $d > 2\alpha$ the spatial structure is complicated ($\mathcal{S}'(\mathbb{R}^d)$ -valued) and the temporal structure is simple (with stationary independent increments). A salient feature of these results is the larger size of the fluctuations at the critical dimension $d = 2\alpha$ ($(T \log T)^{1/2}$ instead of $T^{1/2}$). This phenomenon is known to occur in several models of this type, with or without branching, and related superprocesses (see e.g. Cox and Griffeath, 1984, 1985; Dawson et al, 2001; Deuschel and Wang, 1994; Hong, 2004; Iscoe, 1986). However, no tightness proofs have been given for the rescaled occupation time fluctuations of branching systems, except in the case of intermediate dimensions in BGT.

The above ranges of the parameters are the only ones for which it makes sense to consider fluctuations of the occupation time. See Remark 2.3(d) for the cases $d \leq \alpha$.

Concerning the proofs, the main difference with BGT is in the tightness. In the case $\alpha < d < 2\alpha$ it is relatively simple; it follows from the covariance formula of the empirical process. In the cases $d = 2\alpha$ and $d > 2\alpha$ this formula is not used and a completely new approach is needed. Fourth moments are estimated with the use of a space-time random field method introduced in Bojdecki et al (1986). This method was applied in BGT for the proof of uniqueness and identification of limits (see the introduction of BGT for a general description of this approach). It is noteworthy that the space-time method has turned out to be useful for both purposes in this paper (identification of limits and tightness).

We also present analogous results for the system without branching, obtaining a similar change in the spatial vs. temporal behaviors, the critical dimension being $d = \alpha$ instead of $d = 2\alpha$. In particular, the case $d < \alpha$ leads to long-range dependence (Bojdecki et al, 2004a, 2004b), represented by fractional Brownian motion with covariance

$$\frac{1}{2}(s^h + t^h - |s-t|^h), \quad s, t \geq 0,$$

where $h = 2 - d/\alpha$. This model was considered by Deuschel and Wang (1994) in the case $\alpha = 2$ with different methods that are specific for Brownian motion.

In Section 2 we state the results and in Section 3 we give the proofs.

2. Convergence theorems

We recall the description of the particle system (see BGT for more details). The particles move independently in \mathbb{R}^d according to a symmetric α -stable Lévy process ($0 < \alpha \leq 2$) and undergo critical binary branching (i.e., 0 or 2 particles with probability 1/2 each case) at rate V . Note that the case $V = 0$ corresponds to the system without branching. Let N_t denote the empirical measure of the system at time t . For N_0 we take a Poisson random field with Lebesgue intensity measure λ . We will use the same notation as in BGT. The occupation time fluctuation process $(X_T(t))_{t \geq 0}$ is given by

$$\langle X_T(t), \varphi \rangle = \frac{T}{F_T} \int_0^t (\langle N_{Ts}, \varphi \rangle - \langle \lambda, \varphi \rangle) ds, \quad \varphi \in \mathcal{S}(\mathbb{R}^d) \quad (2.1)$$

(this is clearly the same as (1.1)), where F_T is a norming to be chosen.

In the following theorems and throughout we use the Fourier transform defined as $\widehat{\varphi}(z) = \int_{\mathbb{R}^d} e^{ix \cdot z} \varphi(x) dx$, $z \in \mathbb{R}^d$, where \cdot denotes the scalar product in \mathbb{R}^d .

Theorem 2.1 *For the system without branching ($V = 0$),*

(a) *if $d > \alpha$ and $F_T = T^{1/2}$, we have $X_T \Rightarrow W_0^{(\alpha)}$ in $C([0, \tau], \mathcal{S}'(\mathbb{R}^d))$ as $T \rightarrow \infty$ for any $\tau > 0$, where $(W_0^{(\alpha)}(t))_{t \geq 0}$ is a centered Gaussian $\mathcal{S}'(\mathbb{R}^d)$ -process with covariance function*

$$\text{Cov}(\langle W_0^{(\alpha)}(s), \varphi_1 \rangle, \langle W_0^{(\alpha)}(t), \varphi_2 \rangle) = (s \wedge t) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{2}{|z|^\alpha} \widehat{\varphi}_1(z) \overline{\widehat{\varphi}_2(z)} dz, \quad \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d); \quad (2.2)$$

(b) *if $d = \alpha$ and $F_T = (T \log T)^{1/2}$, we have*

$$X_T \Rightarrow C_d \lambda \beta \text{ as } T \rightarrow \infty, \quad (2.3)$$

where $\beta = (\beta_t)_{t \geq 0}$ is a standard Brownian motion in \mathbb{R} and

$$C_d = \left(2^{d-2} \pi^{d/2} d \Gamma\left(\frac{d}{2}\right) \right)^{-1/2}. \quad (2.4)$$

Theorem 2.2 *For the branching system ($V > 0$),*

(a) *if $d > 2\alpha$ and $F_T = T^{1/2}$, we have $X_T \Rightarrow W^{(\alpha)}$ in $C([0, \tau], \mathcal{S}'(\mathbb{R}^d))$ as $T \rightarrow \infty$ for any $\tau > 0$, where $(W^{(\alpha)}(t))_{t \geq 0}$ is a centered Gaussian $\mathcal{S}'(\mathbb{R}^d)$ -process with covariance function*

$$\text{Cov}(\langle W^{(\alpha)}(s), \varphi_1 \rangle, \langle W^{(\alpha)}(t), \varphi_2 \rangle) = (s \wedge t) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\frac{2}{|z|^\alpha} + \frac{V}{|z|^{2\alpha}} \right) \widehat{\varphi}_1(z) \overline{\widehat{\varphi}_2(z)} dz, \quad \varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d); \quad (2.5)$$

(b) *if $d = 2\alpha$ and $F_T = (T \log T)^{1/2}$, we have*

$$X_T \Rightarrow V^{1/2} C_d \lambda \beta \text{ as } T \rightarrow \infty, \quad (2.6)$$

where $\beta = (\beta_t)_{t \geq 0}$ and C_d are as in Theorem 2.1.

Remark 2.3 (a) More explicit forms of the covariances (2.2) and (2.5) can be given using the formula (see e.g. Gelfand and Shilov, 1964, p. 194)

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\widehat{\varphi}_1(z)\overline{\widehat{\varphi}_2(z)}}{|z|^\gamma} dz = \frac{\Gamma(\frac{d-\gamma}{2})}{2\gamma\pi^{d/2}\Gamma(\frac{\gamma}{2})} \int_{\mathbb{R}^{2d}} \frac{\varphi_1(x)\varphi_2(y)}{|x-y|^{d-\gamma}} dx dy, \quad 0 < \gamma < d. \quad (2.7)$$

(b) The limit processes in Theorem 2.1(a) and 2.2(a) are homogeneous (in space and time) $\mathcal{S}'(\mathbb{R}^d)$ -Wiener processes; in particular they are continuous Gaussian with stationary independent increments. Note that the limit in Theorem 2.2(a) is the sum of two independent $\mathcal{S}'(\mathbb{R}^d)$ -Wiener processes, the first one being the same as in Theorem 2.1(a). Thus, the limit in Theorem 2.2(a) has two parts, one coming from the free (independent) motion of the particles, and the other one incorporating the effect of the branching, while for $\alpha < d < 2\alpha$ the branching had a dominating effect (BGT). $\mathcal{S}'(\mathbb{R}^d)$ -Wiener processes and Gaussian random fields with covariances of the form (2.7) have appeared in several contexts, e.g., renormalization limits of random evolutions (Dawson and Salehi, 1980), occupation time fluctuation limits of two-level branching systems (Dawson et al, 2001), self-intersection local times and related divergence results for $\mathcal{S}'(\mathbb{R}^d)$ -Gaussian processes (Bojdecki and Gorostiza, 1999, Talarczyk, 2001a, 2001b), invariant measures of $\mathcal{S}'(\mathbb{R}^d)$ -Ornstein-Uhlenbeck processes (Bojdecki and Jakubowski, 1999), stochastic wave equations (Dalang, 1999, Dalang and Mueller, 2003).

(c) The fact that the norming in Theorem 2.2(a) is the “classical” one (i.e., as in the classical central limit theorem), is intuitively understood because under the condition $d > 2\alpha$ (which corresponds to strong transience of the α -stable process), the clans (i.e., families of particles with eventually backwards coalescing paths) independently occupy any given ball only during a finite random amount of time each. This behavior is studied in Stöckl and Wakolbinger (1994) for the case $\alpha = 2$ under equilibrium condition. If instead of the Poisson (λ) initial condition, the branching system is started off from an equilibrium state (which exists for $d > \alpha$, Gorostiza and Wakolbinger, 1991), or from a random N_0 which is transported by the α -stable semigroup \mathcal{T}_t to λ as $t \rightarrow \infty$ (Gorostiza and Wakolbinger, 1992, 1994), we expect that the results of Theorem 2.2 also hold.

(d) For the system without branching the results are complete (for all values of d and α). To complete the picture for the branching system (in addition to the results for the intermediate dimensions $\alpha < d < 2\alpha$, where long-range dependence appears, BGT) it remains to consider the cases $d \leq \alpha$. For $d = \alpha$ it is possible to prove a limit theorem for the rescaled occupation time process of the empirical measure process, i.e.,

$$Y_T(t) = \frac{1}{T} \int_0^{Tt} N_s ds, \quad t \geq 0,$$

which is the analogue of Theorem 3 of Iscoe (1986) for super-Brownian motion in dimension $d = 2$ (see also Fleischmann and Gärtner 1986). The limit process for the branching particle system with $\alpha = 2$ coincides with the case of super-Brownian motion. We only state the result (see Talarczyk, 2004, for the proof):

$Y_T \Rightarrow \lambda \xi$ as $T \rightarrow \infty$, where $\xi = (\xi_t)_{t \geq 0}$ is a strictly positive (for $t > 0$) increasing process with finite-dimensional distributions determined by the Laplace transform (which can be obtained from the Laplace transform of the corresponding space-time random field) given by

$$E \exp \left\{ - \sum_{i=1}^k \theta_i \xi_{t_i} \right\} = \exp \{ - \langle \lambda, v(t_k) \rangle \},$$

for $0 \leq t_1 \leq \dots \leq t_k, \theta_1, \dots, \theta_k \geq 0$, where $v(x, t)$ is the mild solution of

$$\frac{\partial}{\partial t} v(t) = \Delta_\alpha v(t) - \frac{V}{2} v(t)^2 + \psi(t) \delta_0, \quad 0 < t < t_k,$$

and

$$\psi(t) = \sum_{i=1}^{k-1} \theta_i \mathbb{1}_{[t_k - t_i, t_k]}(t) + \theta_k,$$

($\Delta_\alpha \equiv -(-\Delta)^{\alpha/2}$ is the infinitesimal generator of the α -stable process).

The result for the fluctuation X_T with norming $F_T = T$ is obtained by subtracting the deterministic process λt . For $d < \alpha$ it is known that $Y_T(1) \rightarrow 0$ (the null measure) a.s. as $T \rightarrow \infty$ (this follows from the persistence/extinction dichotomy, Gorostiza and Wakolbinger, 1991).

(e) The results of Theorem 2.2 should be the same for any critical finite variance branching law with V multiplied by the second factorial moment of the law (because the formula for the covariance of the empirical measure process, i.e. formula (3.1) of BGT, only involves this change; see e.g. Gorostiza, 1983). Binary branching simplifies the proofs. This observation applies also for Theorem 2.2 of BGT.

(f) Hong (2004) proved weak convergence of finite-dimensional distributions for the analogue of Theorem 2.2 in the context of superprocesses (which is easier) with a fixed test function, but not the tightness. Theorem 2.2 implies weak convergence of finite-dimensional distributions with any test functions at different times.

3. Proofs

We first recall some formulas involving Fourier transforms that will be used below (φ_1 , and φ_2 are functions from \mathbb{R}^d to \mathbb{R} , bounded and integrable).

$$\int_{\mathbb{R}^d} \varphi_1(x) \varphi_2(x) dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\varphi}_1(z) \overline{\widehat{\varphi}_2(z)} dz \quad (3.1)$$

(Plancherel formula),

$$\int_{\mathbb{R}^d} \varphi_1(x) \varphi_2(x) d\mu(x) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^{2d}} \widehat{\varphi}_1(z) \widehat{\varphi}_2(z') \overline{\widehat{\mu}(z+z')} dz dz' \quad (3.2)$$

for any finite measure μ ,

$$\widehat{\mathcal{T}_t \varphi_1}(z) = e^{-t|z|^\alpha} \widehat{\varphi}_1(z), \quad (3.3)$$

where \mathcal{T}_t denotes the α -stable semigroup.

We will write C, C_1 , etc. for generic positive constants, with possible dependencies in parenthesis.

A direct proof of convergence of finite-dimensional distributions seems very difficult in our case. Instead we employ the space-time random field method (Bojdecki et al, 1986).

If $X = (X(t))_{t \in [0, \tau]}$ is a continuous $\mathcal{S}'(\mathbb{R}^d)$ -process, we define a random element \tilde{X} of $\mathcal{S}'(\mathbb{R}^{d+1})$ by

$$\langle \tilde{X}, \Phi \rangle = \int_0^\tau \langle X(t), \Phi(t, \cdot) \rangle dt, \quad \Phi \in \mathcal{S}(\mathbb{R}^{d+1}). \quad (3.4)$$

In order to prove all assertions of the theorems it suffices to show

(i) $\langle \tilde{X}_T, \Phi \rangle \Rightarrow \langle \tilde{X}, \Phi \rangle$ as $T \rightarrow \infty$, $\Phi \in \mathcal{S}(\mathbb{R}^{d+1})$, where X is the corresponding limit process, and

(ii) $\{\langle X_T; \varphi \rangle; T \geq 2\}$ is tight in $C([0, \tau], \mathbb{R})$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

where in (ii) we also use the theorem of Mitoma (1983).

As explained in BGT, (i) will be proved if we show that

$$\lim_{T \rightarrow \infty} E \exp\{-\langle \tilde{X}_T, \Phi \rangle\} = \exp\left\{\frac{1}{2} \int_0^\tau \int_0^\tau \text{Cov}(\langle X(s), \Phi(s, \cdot) \rangle, \langle X(t), \Phi(t, \cdot) \rangle) ds dt\right\} \quad (3.5)$$

for each non-negative $\Phi \in \mathcal{S}(\mathbb{R}^{d+1})$.

We assume without loss of generality that $\tau = 1$. We give only the proof of Theorem 2.2, since the proof of Theorem 2.1 is analogous but simpler.

Proof of Theorem 2.2

To avoid cumbersome notation we prove (3.5) for Φ of the form $\Phi(t, x) = \varphi(x)\psi(t)$, $\varphi, \psi \geq 0$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in \mathcal{S}(\mathbb{R})$. So, we fix φ, ψ and denote

$$\varphi_T(x) = \frac{1}{F_T} \varphi(x), \quad \chi(t) = \int_t^1 \psi(u) du, \quad \chi_T(t) = \chi\left(\frac{t}{T}\right). \quad (3.6)$$

Repeating the argument in BGT (see (3.10)-(3.23) therein) we obtain the Laplace functional of \tilde{X}_T defined by (3.4):

$$\begin{aligned} E \exp\{-\langle \tilde{X}_T, \Phi \rangle\} &= \exp\left\{\int_0^T \int_{\mathbb{R}^d} \varphi_T(x) \chi_T(T-u) v_{\varphi_T, \chi_T}(x, T-u, u) dx du \right. \\ &\quad \left. + \frac{V}{2} \int_0^T \int_{\mathbb{R}^d} (v_{\varphi_T, \chi_T}(x, T-u, u))^2 dx du\right\}, \end{aligned} \quad (3.7)$$

where

$$v_{\varphi, \chi}(x, r, t) = 1 - E \exp\left\{-\int_0^t \langle N_s^x, \varphi \rangle \chi(r+s) ds\right\}, \quad (3.8)$$

and N_s^x is the empirical measure of the particle system with initial condition $N_0^x = \delta_x$. Moreover, by the Feynman-Kac formula we know that $v_{\varphi, \chi}$ satisfies

$$\begin{aligned} v_{\varphi, \chi}(x, r, t) &= \int_0^t \mathcal{T}_{t-s} \left[\varphi(\cdot) \chi(r+t-s) (1 - v_{\varphi, \chi}(\cdot, r+t-s, s)) \right. \\ &\quad \left. - \frac{V}{2} (v_{\varphi, \chi}(\cdot, r+t-s, s))^2 \right] (x) ds, \end{aligned} \quad (3.9)$$

hence

$$0 \leq v_{\varphi, \chi}(x, r, t) \leq \int_0^t \mathcal{T}_{t-s} \varphi(x) \chi(r+t-s) ds. \quad (3.10)$$

We rewrite (3.7) as

$$E \exp\{-\langle \tilde{X}_T, \Phi \rangle\} = \exp\left\{\frac{V}{2} (I_1(T) + I_2(T)) + I_3(T)\right\}, \quad (3.11)$$

where

$$I_1(T) = \int_0^T \int_{\mathbb{R}^d} \left(\int_0^u \mathcal{T}_{u-s} \varphi_T(x) \chi_T(T-s) ds \right)^2 dx du, \quad (3.12)$$

$$I_2(T) = \int_0^T \int_{\mathbb{R}^d} \left[(v_{\varphi_T, \chi_T}(x, T-u, u))^2 - \left(\int_0^u \mathcal{T}_{u-s} \varphi_T(x) \chi_T(T-s) ds \right)^2 \right] dx du, \quad (3.13)$$

$$I_3(T) = \int_0^T \int_{\mathbb{R}^d} \varphi_T(x) \chi_T(T-u) v_{\varphi_T, \chi_T}(x, T-u, u) dx du. \quad (3.14)$$

For part (a) of the theorem we will prove

$$I_1(T) \rightarrow \frac{1}{(2\pi)^d} \int_0^1 \int_0^1 (r \wedge r') \psi(r) \psi(r') dr' dr \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^{2\alpha}} dz, \quad (3.15)$$

$$I_2(T) \rightarrow 0 \quad (3.16)$$

$$I_3(T) \rightarrow \frac{1}{(2\pi)^d} \int_0^1 \int_0^1 (r \wedge r') \psi(r) \psi(r') dr' dr \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^\alpha} dz. \quad (3.17)$$

as $T \rightarrow \infty$, which, taking into account (2.5) yields (3.5).

Using (3.6) and making obvious substitutions we have

$$\begin{aligned} I_1(T) &= \frac{T^3}{F_T^2} \int_0^1 \int_u^1 \int_u^1 \int_{\mathbb{R}^d} \mathcal{T}_{T(s-u)} \varphi(x) \mathcal{T}_{T(s'-u)} \varphi(x) \chi(s) \chi(s') dx ds ds' du \\ &= \frac{T^2}{(2\pi)^d} \int_0^1 \int_u^1 \int_u^1 \int_{\mathbb{R}^d} e^{-T(s-u)|z|^\alpha} e^{-T(s'-u)|z|^\alpha} |\widehat{\varphi}(z)|^2 \chi(s) \chi(s') dz ds ds' du \\ &= \frac{T^2}{(2\pi)^d} \int_0^1 \int_0^1 \int_0^{r \wedge r'} \int_{\mathbb{R}^d} \int_u^r e^{-T(s-u)|z|^\alpha} ds \int_u^{r'} e^{-T(s'-u)|z|^\alpha} ds' |\widehat{\varphi}(z)|^2 dz du \psi(r) \psi(r') dr dr', \end{aligned} \quad (3.18)$$

where in the second equality we used (3.1) and (3.3), and for the last one we put $\chi(s) = \int_s^1 \psi(r) dr$, $\chi(s') = \int_{s'}^1 \psi(r') dr'$, and we changed the order of integration. It is now easy to see that (3.15) indeed holds.

Next, using (3.9) and (3.10) in the same manner as in BGT (see (3.35)-(3.42) therein) we have from (3.13)

$$0 \leq -I_2(T) \leq 2J_1(T) + VJ_2(T), \quad (3.19)$$

where

$$\begin{aligned} J_1(T) &= \int_0^T \int_0^u \int_0^u \int_0^s \int_{\mathbb{R}^d} \mathcal{T}_{u-s'} \varphi_T(x) \mathcal{T}_{u-s} [\varphi_T(\cdot) \mathcal{T}_{s-r} \varphi_T(\cdot)](x) \chi_T(T-s') \\ &\quad \times \chi_T(T-s) \chi_T(T-r) dx dr ds ds' du, \end{aligned} \quad (3.20)$$

$$\begin{aligned} J_2(T) &= \int_0^T \int_0^u \int_0^u \int_0^s \int_0^s \int_{\mathbb{R}^d} \mathcal{T}_{u-s'} \varphi_T(x) \mathcal{T}_{u-s} [\mathcal{T}_{s-r} \varphi_T(\cdot) \mathcal{T}_{s-r'} \varphi_T(\cdot)](x) \chi_T(T-s') \\ &\quad \times \chi_T(T-r) \chi_T(T-r') dx dr dr' ds ds' du. \end{aligned} \quad (3.21)$$

By (3.6) and boundedness of χ , after obvious substitutions we obtain

$$J_1(T) \leq C \frac{T^4}{F_T^3} \int_0^1 \int_0^u \int_0^u \int_0^s \int_{\mathbb{R}^d} \mathcal{T}_{T(u-s')} \varphi(x) \mathcal{T}_{T(u-s)} [\varphi \mathcal{T}_{T(s-r)} \varphi](x) dx dr ds ds' du.$$

We now use the self-adjointness of $\mathcal{T}_{T(u-s)}$, formula (3.2) with $\mu(dx) = \varphi(x)dx$, and (3.3) to obtain that the right-hand side of this inequality is equal to

$$C \frac{T^{5/2}}{(2\pi)^{2d}} \int_0^1 \int_0^u \int_0^u \int_0^s \int_{\mathbb{R}^{2d}} e^{-T(2u-s'-s)|z|^\alpha} e^{-T(s-r)|z'|^\alpha} \widehat{\varphi}(z) \widehat{\varphi}(z') \overline{\widehat{\varphi}(z+z')} dz dz' dr ds ds' du. \quad (3.22)$$

We need the following trivial estimate which will be used several times (in both forms of the integral),

$$\int_0^u e^{-T(u-r)|z|^\alpha} dr = \int_{1-u}^1 e^{-T(r-(1-u))|z|^\alpha} dr \leq \frac{1}{T|z|^\alpha}, \quad 0 \leq u \leq 1, \quad z \in \mathbb{R}^d. \quad (3.23)$$

We apply (3.23) in (3.22) first to the integral dr and then to ds, ds' , obtaining (since $\widehat{\varphi}$ is bounded)

$$J_1(T) \leq C_1 T^{-1/2} \int_{\mathbb{R}^{2d}} \frac{|\widehat{\varphi}(z)|}{|z|^{2\alpha}} \frac{|\widehat{\varphi}(z')|}{|z'|^\alpha} dz' dz,$$

hence $J_1(T) \rightarrow 0$ as $T \rightarrow \infty$, because $d > 2\alpha$.

$J_2(T)$ can be estimated in exactly the same manner, the only difference being that now we use (3.2) with $\mu(dx) = \mathcal{T}_{s-r'} \varphi_T(x) dx$, therefore in the final estimate we obtain

$$J_2(T) \leq C_2 T^{-1/2} \int_{\mathbb{R}^{2d}} \frac{|\widehat{\varphi}(z)|}{|z|^{2\alpha}} \frac{|\widehat{\varphi}(z+z')|}{|z+z'|^\alpha} \frac{|\widehat{\varphi}(z')|}{|z'|^\alpha} dz' dz.$$

The latter integral is finite since the function

$$z \mapsto \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z+z')|}{|z+z'|^\alpha} \frac{|\widehat{\varphi}(z')|}{|z'|^\alpha} dz'$$

is bounded, hence $J_2(T) \rightarrow 0$ as $T \rightarrow \infty$. Consequently, (3.16) is proved by (3.19).

We now pass to (3.17). (In BGT $I_3(T) \rightarrow 0$ was easy to obtain; in the present situation we have a non-trivial limit here, hence more work is needed). Using (3.9) we rewrite (3.14) as

$$I_3(T) = I_3'(T) - I_3''(T) - I_3'''(T), \quad (3.24)$$

where

$$I_3'(T) = \int_0^T \int_{\mathbb{R}^d} \varphi_T(x) \chi_T(T-u) \int_0^u \mathcal{T}_{u-s} \varphi_T(x) \chi_T(T-s) ds dx du, \quad (3.25)$$

$$I_3''(T) = \int_0^T \int_{\mathbb{R}^d} \varphi_T(x) \chi_T(T-u) \int_0^u \mathcal{T}_{u-s} [\varphi_T(\cdot) \chi_T(T-s) v_{\varphi_T, \chi_T}(\cdot, T-s, s)](x) ds dx du, \quad (3.26)$$

$$I_3'''(T) = \frac{V}{2} \int_0^T \int_{\mathbb{R}^d} \varphi_T(x) \chi_T(T-u) \int_0^u \mathcal{T}_{u-s} (v_{\varphi_T, \chi_T}(\cdot, T-s, s))^2(x) ds dx du. \quad (3.27)$$

If we compare I_3' to I_1 above (see (3.12), (3.18)), we see that it can be treated analogously, even more easily, and we obtain

$$I_3'(T) \rightarrow \frac{1}{(2\pi)^d} \int_0^1 \int_0^1 (r \wedge r') \psi(r) \psi(r') dr' dr \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^\alpha} dz \quad \text{as } T \rightarrow \infty. \quad (3.28)$$

Next we estimate v_{φ_T, χ_T} in I_3'' using (3.10), and we obtain an expression similar to J_1 above (see (3.20)). We apply the same technique based on (3.2), (3.3) and (3.23) to obtain

$$I_3''(T) \rightarrow 0 \quad \text{as } T \rightarrow \infty. \quad (3.29)$$

Finally, after estimating v_{φ_T, χ_T} in I_3''' with the help of (3.10) we arrive at an expression similar to J_2 above (see (3.21)), and in the same way as before we obtain

$$I_3'''(T) \rightarrow 0 \quad \text{as } T \rightarrow \infty. \quad (3.30)$$

(3.24)-(3.30) prove (3.17), and this completes the proof of (3.5) for part (a) of the theorem. For part (b) we will show

$$I_1(T) \rightarrow C_d^2 \int_0^1 \int_0^1 (r \wedge r') \psi(r) \psi(r') dr dr' \left(\int_{\mathbb{R}^d} \varphi(x) dx \right)^2, \quad (3.31)$$

$$I_2(T) \rightarrow 0, \quad (3.32)$$

$$I_3(T) \rightarrow 0 \quad (3.33)$$

as $T \rightarrow \infty$ (see (2.6), (3.5), (3.7), (3.11)-(3.14)).

We write (3.18), now with $F_T = (T \log T)^{1/2}$, and calculating the integrals ds and ds' in the last expression we arrive at

$$I_1(T) = \frac{1}{(2\pi)^d} \frac{1}{\log T} \int_0^1 \int_0^1 \psi(r) \psi(r') \int_0^{r \wedge r'} \int_{\mathbb{R}^d} \frac{1 - e^{-T(r-u)|z|^{d/2}}}{|z|^{d/2}} \frac{1 - e^{-T(r'-u)|z|^{d/2}}}{|z|^{d/2}} \\ \times |\widehat{\varphi}(z)|^2 dz du dr dr'.$$

Then (3.31) follows by L'Hôpital's rule. Indeed, after differentiating w.r.t. T under the integrals we substitute $T^{2/d}z = z'$, then we have $|\widehat{\varphi}(T^{-2/d}z')|^2 \rightarrow |\widehat{\varphi}(0)|^2 = (\int_{\mathbb{R}^d} \varphi(x) dx)^2$, and

$$\int_{\mathbb{R}^d} \left[(r-u) e^{-(r-u)|z'|^{d/2}} \frac{1 - e^{-(r'-u)|z'|^{d/2}}}{|z'|^{d/2}} + \frac{1 - e^{-(r-u)|z'|^{d/2}}}{|z'|^{d/2}} (r'-u) e^{-(r'-u)|z'|^{d/2}} \right] dz' \\ = (2\pi)^d C_d^2.$$

independently of r, r' and u .

To prove (3.32) we again write (3.19) with J_1, J_2 given by (3.20), (3.21) respectively. J_1 is bounded above by (3.22) with the only difference that now the coefficient before the integrals has the form $CT^{5/2}/(\log T)^{3/2}$.

We now need a slightly more precise (but equally trivial) version of the estimate (3.23):

$$\int_0^u e^{-T(u-r)|z|^\alpha} dr = \int_{1-u}^1 e^{-T(r-(1-u))|z|^\alpha} dr \leq \frac{1 - e^{-T|z|^\alpha}}{T|z|^\alpha}, \quad 0 \leq u \leq 1, \quad z \in \mathbb{R}^d. \quad (3.34)$$

We will also use

$$\sup_{T>2} \left(\frac{1}{\log T} \int_{\mathbb{R}^d} \frac{1 - e^{-T|z|^{d/2}}}{|z|^d} f(z) dz \right) < \infty, \quad (3.35)$$

which holds for any non-negative bounded and integrable function f , and is checked easily with L'Hôpital's rule.

In (3.22) we apply (3.23) to the integral dr , then to ds , and finally (3.34) to ds' . We obtain

$$J_1(T) \leq \frac{C_1}{T^{1/2}(\log T)^{3/2}} \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z')|}{|z'|^{d/2}} dz' \int_{\mathbb{R}^d} \frac{1 - e^{-T|z|^{d/2}}}{|z|^d} |\widehat{\varphi}(z)| dz,$$

hence, using (3.35) we have $J_1(T) \rightarrow 0$ as $T \rightarrow \infty$.

J_2 can be treated in the same way as in the case with $d > 2\alpha$, with (3.23) replaced by (3.34). We obtain

$$J_2(T) \leq \frac{C}{T^{1/2}(\log T)^{3/2}} \int_{\mathbb{R}^{2d}} \frac{(1 - e^{-T|z|^{d/2}})^2}{|z|^d} \frac{1 - e^{-T|z'|^{d/2}}}{|z'|^{d/2}} \frac{1 - e^{-T|z+z'|^{d/2}}}{|z+z'|^{d/2}} \\ \times |\widehat{\varphi}(z)| |\widehat{\varphi}(z')| |\widehat{\varphi}(z+z')| dz' dz.$$

Using the obvious inequality $1 - e^{-x} \leq x^{1/8}$ ($x \geq 0$), this is estimated from above by

$$\frac{C}{(\log T)^{3/2}} \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|}{|z|^{d7/8}} \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z')|}{|z'|^{d7/16}} \frac{|\widehat{\varphi}(z+z')|}{|z+z'|^{d7/16}} dz' dz.$$

The last integral is finite (the function $z' \mapsto |\widehat{\varphi}(z')|/|z'|^{d7/16}$ is square integrable). Hence we conclude that $J_2(T) \rightarrow 0$ as $T \rightarrow \infty$. This completes the proof of (3.32).

The proof of (3.33) is very easy. We use (3.10) and apply the same technique as before, based on (3.1) and (3.3). We omit details.

We pass now to the proof of tightness. By Billingsley (1968), it suffices to show that for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\varphi \geq 0$, we have

$$E(\langle X_T(t), \varphi \rangle - \langle X_T(s), \varphi \rangle)^4 \leq C(\varphi)(t-s)^2, \quad (3.36)$$

$0 \leq s < t \leq 1, T \geq 2$. Indeed, since each $\varphi \in \mathcal{S}(\mathbb{R}^d)$ can be written as $\varphi = \varphi_1 - \varphi_2$, $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, $\varphi_1, \varphi_2 \geq 0$, then (3.36) implies tightness of the processes $\{\langle X_T, \varphi \rangle, T \geq 2\}$ for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, so tightness of $\{X_T, T \geq 2\}$ follows by Mitoma's theorem (1983).

So, we fix $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\varphi \geq 0$ and $s, t \in [0, 1]$, $s < t$. For $n > 2/(t-s)$, let $\psi_n \in \mathcal{S}(\mathbb{R})$ be such that $\text{supp}(\psi_n) \subset [s, s+1/n] \cup [t-1/n, t]$, $\psi_n \leq 0$ on $[s, s+1/n]$ and $\int_s^{s+1/n} \psi_n(u) du = -1$, $\psi_n \geq 0$ on $[t-1/n, t]$ and $\int_{t-1/n}^t \psi_n(u) du = 1$.

As $\psi_n \rightarrow \delta_t - \delta_s$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \langle \widetilde{X}_T, \Phi_n \rangle = \lim_{n \rightarrow \infty} \int_0^1 \langle X_T(u), \varphi \rangle \psi_n(u) du = \langle X_T(t), \varphi \rangle - \langle X_T(s), \varphi \rangle,$$

where $\Phi_n = \varphi \otimes \psi_n$. Hence, by Fatou's lemma, to obtain (3.36) it suffices to show that

$$E \langle \widetilde{X}_T, \Phi_n \rangle^4 \leq C(\varphi)(t-s)^2, \quad (3.37)$$

$n > 2/(t-s), T > 2$.

We write the left-hand side of (3.37) as $\frac{d^4}{d\theta^4} E^{-\theta(\widetilde{X}_T, \Phi_n)}|_{\theta=0}$, and this expression suggests the possibility of using formula (3.7). We apply this formula to $\theta\varphi$ ($\theta \geq 0$) instead of φ , and to $\chi_n(u) = \int_u^1 \psi_n(r) dr$ instead of χ . Observe that

$$\chi_n \in \mathcal{S}(\mathbb{R}), \quad 0 \leq \chi_n \leq \mathbf{1}_{[s,t]}, \quad (3.38)$$

hence (3.37) will be proved if we show that

$$\frac{d^4}{d\theta^4} e^{H(\theta)}|_{\theta=0} \leq C(\varphi)(t-s)^2 \quad (3.39)$$

for each χ satisfying (3.38), where

$$H(\theta) = \theta \int_0^T \int_{\mathbb{R}^d} \varphi_T(x) \chi_T(T-u) v_{\theta\varphi_T, \chi_T}(x, T-u, u) dx du \\ + \frac{V}{2} \int_0^T \int_{\mathbb{R}^d} (v_{\theta\varphi_T, \chi_T}(x, T-u, u))^2 dx du. \quad (3.40)$$

We have

$$\frac{d^4}{d\theta^4}e^H = ((H')^4 + 6(H')^2H'' + 4H'H''' + 3(H'')^2 + H^{IV})e^H.$$

On the other hand, $v_{0,\chi_T} \equiv 0$ (see (3.8)), hence $H(0) = 0, H'(0) = 0$, therefore

$$\frac{d^4}{d\theta^4}e^{H(\theta)}|_{\theta=0} = H^{IV}(0) + 3(H''(0))^2.$$

Consequently, to obtain (3.39) is suffices to prove that

$$|H''(0)| \leq C(\varphi)(t-s) \quad (3.41)$$

and

$$|H^{IV}(0)| \leq C(\varphi)(t-s)^2. \quad (3.42)$$

It will be convenient to denote

$$v(\theta) = v(\theta)(x, T-u, u) = v_{\theta\varphi_T, \chi_T}(x, T-u, u).$$

We then have, from (3.40),

$$H''(0) = 2 \int_0^T \int_{\mathbb{R}^d} \varphi_T(x) \chi_T(T-u) v'(0) dx du + V \int_0^T \int_{\mathbb{R}^d} (v'(0))^2 dx du \quad (3.43)$$

and

$$\begin{aligned} H^{IV}(0) &= 4 \int_0^T \int_{\mathbb{R}^d} \varphi_T(x) \chi_T(T-u) v'''(0) dx du + 4V \int_0^T \int_{\mathbb{R}^d} v'(0) v'''(0) dx du \\ &\quad + 3V \int_0^T \int_{\mathbb{R}^d} (v''(0))^2 dx du. \end{aligned} \quad (3.44)$$

By (3.9) we have

$$v'(0)(x, T-u, u) = \int_0^u \mathcal{T}_{u-u_1} \varphi_T(x) \chi_T(T-u_1) du_1, \quad (3.45)$$

$$\begin{aligned} v''(0)(x, T-u, u) &= -2 \int_0^u \mathcal{T}_{u-u_1} [\varphi_T(\cdot) v'(\cdot)(\cdot, T-u_1, u_1)](x) \chi_T(T-u_1) du_1 \\ &\quad - V \int_0^u \mathcal{T}_{u-u_1} [v'(0)(\cdot, T-u_1, u_1)]^2(x) du_1, \end{aligned} \quad (3.46)$$

$$\begin{aligned} v'''(0)(x, T-u, u) &= -3 \int_0^u \mathcal{T}_{u-u_1} [\varphi_T(\cdot) v''(\cdot)(\cdot, T-u_1, u_1)](x) \chi_T(T-u_1) du_1 \\ &\quad - 3V \int_0^u \mathcal{T}_{u-u_1} [v'(0)(\cdot, T-u_1, u_1) v''(\cdot, T-u_1, u_1)](x) du_1. \end{aligned} \quad (3.47)$$

Before we proceed, let us write down two estimates which follow immediately from (3.38) and which will be used several times.

$$\int_u^1 e^{-T(r-u)|z|^\alpha} \chi(r) dr \leq t-s, \quad 0 \leq u \leq 1, \quad z \in \mathbb{R}^d, \quad (3.48)$$

$$\int_0^1 \int_u^1 e^{-T(r-u)|z|^\alpha} \chi(r) dr du \leq \frac{t-s}{T|z|^\alpha}, \quad z \in \mathbb{R}^d. \quad (3.49)$$

We will prove tightness for part (a) of the theorem. In order to prove (3.41) we estimate the two terms on the right-hand side of (3.43) separately. Let us consider for instance the second term (omitting the irrelevant coefficient V). By (3.45) we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} (v'(0))^2 dx du \\ &= \int_0^T \int_0^u \int_0^u \int_{\mathbb{R}^d} \mathcal{T}_{u-u_1} \varphi_T(x) \mathcal{T}_{u-u_2} \varphi_T(x) dx \chi\left(1 - \frac{u_1}{T}\right) \chi\left(1 - \frac{u_2}{T}\right) du_1 du_2 du \\ &= T^2 \int_0^1 \int_{1-u}^1 \int_{1-u}^1 \int_{\mathbb{R}^d} \mathcal{T}_{T(u_1-(1-u))} \varphi(x) \mathcal{T}_{T(u_2-(1-u))} \varphi(x) dx \chi(u_1) \chi(u_2) du_1 du_2 du \end{aligned}$$

(by (3.1), (3.3))

$$\begin{aligned} &= \frac{T^2}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{\varphi}(z)|^2 \int_0^1 \int_{1-u}^1 e^{-T(u_1-(1-u))|z|^\alpha} \chi(u_1) du_1 \int_{1-u}^1 e^{-T(u_2-(1-u))|z|^\alpha} \chi(u_2) du_2 du dz \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|^2}{|z|^{2\alpha}} dz (t-s), \end{aligned}$$

by (3.23) and (3.49). It suffices to observe that the last integral is finite since $d > 2\alpha$.

The first summand on the right hand side of (3.43) is estimated similarly, even more easily.

The proof of (3.42) requires much more work. In fact, looking at (3.44)-(3.47) it is clear that as many as 11 terms have to be estimated. Fortunately, the idea for treating them remains the same; it is based on the Fourier transform technique and on estimates (3.23), (3.48) and (3.49). Let us consider the term which is perhaps most impressive, i.e., the one coming from the summand involving $v'(0)v'''(0)$ in (3.44), where in $v'''(0)$ we take the expression with $v'(0)v''(0)$ (see (3.47)), and in $v''(0)$ we consider the second summand (see (3.46)). Omitting numerical coefficients and powers of V we have

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \int_0^u \mathcal{T}_{u-u_1} \varphi_T(x) \chi_T(T-u_1) du_1 \int_0^u \mathcal{T}_{u-u'_1} \left[\int_0^{u'_1} \mathcal{T}_{u'_1-u_2} \varphi_T(\cdot) \chi_T(T-u_2) du_2 \right. \\ & \times \left. \int_0^{u'_1} \mathcal{T}_{u'_1-u'_2} \left(\int_0^{u'_2} \mathcal{T}_{u'_2-u_3} \varphi_T \chi_T(T-u_3) du_3 \right)^2 (\cdot) du'_2 \right] (x) du'_1 dx du \\ &= \frac{1}{T^2} \int_0^T \int_0^u \int_0^u \int_0^{u'_1} \int_0^{u'_1} \int_0^{u'_2} \int_0^{u'_2} \int_{\mathbb{R}^d} \mathcal{T}_{u-u_1} \varphi(x) \mathcal{T}_{u-u'_1} \left[\mathcal{T}_{u'_1-u_2} \varphi(\cdot) \right. \\ & \times \left. \mathcal{T}_{u'_1-u'_2} \left(\mathcal{T}_{u'_2-u_3} \varphi \mathcal{T}_{u'_2-u'_3} \varphi \right) (\cdot) \right] (x) dx \chi\left(1 - \frac{u_1}{T}\right) \chi\left(1 - \frac{u_2}{T}\right) \chi\left(1 - \frac{u_3}{T}\right) \chi\left(1 - \frac{u'_3}{T}\right) \\ & \quad du_3 du'_3 du_2 du'_2 du_1 du'_1 du. \end{aligned} \tag{3.50}$$

We transform the integral dx using, consecutively, the self-adjointness of $\mathcal{T}_{u-u'_1}$, formulas (3.2) with $\mu(dx) = \mathcal{T}_{u'_1-u_2} \varphi(x) dx$ and (3.3), and the fact that the Fourier transform of a product is the convolution of the transforms. We obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \dots dx &= C \int_{\mathbb{R}^{3d}} e^{-(2u-u_1-u'_1)|z|^\alpha} e^{-(u'_1-u'_2)|z'|^\alpha} e^{-(u'_2-u_3)|z'-w|^\alpha} e^{-(u'_2-u'_3)|w|^\alpha} \\ & \quad \times e^{-(u'_1-u_2)|z+z'|^\alpha} \widehat{\varphi}(z) \widehat{\varphi}(z'-w) \widehat{\varphi}(w) \overline{\widehat{\varphi}(z+z')} dw dz' dz. \end{aligned} \tag{3.51}$$

We put this back into the right hand side of (3.50), bring the space integrals outside and look at the time integrals.

After obvious substitutions we have

$$\begin{aligned}
& \frac{1}{T^2} \int_0^T \int_0^u \int_0^u \int_0^{u_1} \int_0^{u_1} \int_0^{u_2} \int_0^{u_2} \dots du_3 du_3' du_2 du_2' du_1 du_1' du \\
&= T^5 \int_0^1 \int_r^1 \int_r^1 \int_{r_1'}^1 \int_{r_1'}^1 \int_{r_2'}^1 \int_{r_2'}^1 e^{-T(r_1+r_1'-2r)|z|^\alpha} e^{-T(r_2'-r_1')|z'|^\alpha} e^{-T(r_3-r_2')|z'-w|^\alpha} \\
&\quad \times e^{-T(r_3'-r_2')|w|^\alpha} e^{-T(r_2-r_1')|z+z'|^\alpha} \chi(r_1)\chi(r_2)\chi(r_3)\chi(r_3') dr_3 dr_3' dr_2 dr_2' dr_1 dr_1' dr \\
&= T^5 \int_0^1 \int_r^1 e^{-T(r_1-r)|z|^\alpha} \chi(r_1) dr_1 \int_r^1 e^{-T(r_1'-r)|z|^\alpha} \int_{r_1'}^1 e^{-(r_2-r_1')|z+z'|^\alpha} \chi(r_2) dr_2 \\
&\quad \times \int_{r_1'}^1 e^{-T(r_2'-r_1')|z'|^\alpha} \int_{r_2'}^1 e^{-T(r_3-r_2')|z'-w|^\alpha} \chi(r_3) dr_3 \int_{r_2'}^1 e^{-T(r_3'-r_2')|w|^\alpha} \chi(r_3') dr_3' dr_2' dr_1' dr.
\end{aligned} \tag{3.52}$$

We apply (3.23), consecutively, to the integrals dr_3', dr_3, dr_2' , then (3.48) to dr_2 and (3.23) once again to dr_1' , and finally (3.49) to $dr_1 dr$. Consequently, taking into account (3.51) we obtain that the left-hand side of (3.50) is estimated from above by

$$\begin{aligned}
& C \int_{\mathbb{R}^{3d}} \frac{|\widehat{\varphi}(z)|}{|z|^{2\alpha}} \frac{1}{|z'|^\alpha} \frac{|\widehat{\varphi}(z'-w)|}{|z'-w|^\alpha} \frac{|\widehat{\varphi}(w)|}{|w|^\alpha} |\widehat{\varphi}(z+z')| dw dz' dz (t-s)^2 \\
&\leq C_1 \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z)|}{|z|^{2\alpha}} dz \int_{\mathbb{R}^d} \frac{1}{|z'|^\alpha} \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z'-w)|}{|z'-w|^\alpha} \frac{|\widehat{\varphi}(w)|}{|w|^\alpha} dw dz' (t-s)^2.
\end{aligned}$$

It suffices now to observe that all integrals are finite since $d > 2\alpha$. Note that $f(w) = |\widehat{\varphi}(w)|/|w|^\alpha$ belongs to $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, hence $f * f$ is bounded and integrable.

The remaining terms in (3.44) can be estimated in a similar way, thus yielding (3.42). This completes the proof of tightness for part (a) of the theorem.

The proof for part (b) goes along the same lines. Only the final estimates have to be slightly more precise. Let us look for example at the counterpart of (3.52). It looks exactly the same, with the only difference that an additional factor $1/(\log T)^2$ appears before the integrals. Now we apply, consecutively, (3.23) to the integrals dr_3', dr_3 , then (3.34) to dr_2' , (3.48) to dr_2 , (3.34) once again to dr_1' , and finally (3.49) to $dr_1 dr$. We obtain that the left-hand side of (3.50) is estimated from above by

$$\begin{aligned}
& C \frac{1}{\log T} \int_{\mathbb{R}^d} \frac{1 - e^{-T|z|^{d/2}}}{|z|^d} |\widehat{\varphi}(z)| dz \frac{1}{\log T} \int_{\mathbb{R}^d} \frac{1 - e^{-T|z'|^{d/2}}}{|z'|^{d/2}} \\
&\quad \times \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z'-w)|}{|z'-w|^{d/2}} \frac{|\widehat{\varphi}(w)|}{|w|^{d/2}} dw dz' (t-s)^2.
\end{aligned}$$

We now use (3.35) twice; for the first factor with $f = |\widehat{\varphi}|$ and for the second one with

$$f(z') = |z'|^{d/2} \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z'-w)|}{|z'-w|^{d/2}} \frac{|\widehat{\varphi}(w)|}{|w|^{d/2}} dw.$$

Observe that this f is indeed bounded and integrable since

$$f(z') \leq C \left(\int_{\mathbb{R}^d} |\widehat{\varphi}(z'-w)| \frac{|\widehat{\varphi}(w)|}{|w|^{d/2}} dw + \int_{\mathbb{R}^d} \frac{|\widehat{\varphi}(z'-w)|}{|z'-w|^{d/2}} |\widehat{\varphi}(w)| dw \right).$$

Arguing similarly for all the remaining cases we obtain (3.41) and (3.42), and tightness is proved.

The proof of Theorem 2.2 is complete. □

The proof of Theorem 2.1 is analogous but easier, since the fundamental formulas (3.7) and (3.9) have simpler forms (with $V = 0$). We also note that the proof of tightness can be made more directly in this case (by no means trivially, though), i.e., without the use of the space-time method, since $E\langle X_T, \varphi \rangle^4$ can be calculated explicitly.

Acknowledgment During the writing of this paper T.B. and L.G.G. were visiting the Institute of Mathematics, National University of Mexico (UNAM), and A.T. was visiting the University of British Columbia (Vancouver, Canada) with the support of an NSERC grant. The authors are grateful for the hospitality of these institutions.

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