Path Properties of Forty Years of Research in Probability and Statistics: In Conversation with Miklós Csörgő

Edited by L. Horváth and B. Szyszkowicz

Prologue

We quote from our Preface to the volume: *Asymptotic Methods in Stochastics: Festschrift for Miklós Csörgő* (L. Horváth, B. Szyszkowicz, Eds.), Fields Institute Communications, Volume 44, AMS 2004.

> The papers in this volume reflect the wide ranging interest of Miklós in Probability and Statistics, and nearly all of them are connected to his research. The editors also have a 69 page résumé of his work over the past forty or so years, titled Path Properties of Forty Years of Research in Probability and Statistics: In Conversation with Miklós Csörgő. This article, together with Miklós's list of publications is available as No. 400–2004 of the Technical Report Series of LRSP. It can also be accessed on the LRSP website: `www.lrsp.carleton.ca`, as well as on the Fields Institute website: `www.fields.utoronto.ca/publications/supplements/`. Unfortunately, due to space limitations, we could not include this résumé with its 311 references and Miklós's list of publications in this collection.

For the sake of better connecting to the above volume, we are reprinting the full text of our PREFACE to it in this No. 400 - 2004 of the Technical Report Series of LRSP.

Our long résumé of Miklós Csörgő's work over the past forty or so years is meant to read as if it were an expository survey paper. On occasions there are also references made to papers in the Festschrift volume. It is hoped that when viewed together with the volume, our résumé will also contribute to, and thus enhance, the cohesion of the whole volume as an expository research monograph. References in our résumé that are made to papers in Publications of Miklós Csörgő, as summarized in this report, are designated by bold-face numbers in square brackets.

Lajos Horváth
Barbara Szyszkowicz

Preface

ICAMS'02, an International Conference on Asymptotic Methods in Stochastics was organized and held in honour of the work of Miklós Csörgő on the occasion of his 70th birthday at Carleton University, Ottawa, Canada, 23–25 May 2002. The conference was hosted and sponsored by the Laboratory for Research in Statistics and Probability (LRSP), Carleton University–University of Ottawa, the School of Mathematics and Statistics, Carleton University, and co-sponsored by The Fields Institute for Research in Mathematical Sciences. This international meeting was a smaller version of an earlier conference, ICAMPS'97 (International Conference on Asymptotic Methods in Probability and Statistics), that was held at Carleton University in July 1997. For the proceedings volume of the latter conference we refer to [1].

We are pleased to publish the proceedings of ICAMS'02 in Fields Institute Communications by AMS, and it is our pleasure to dedicate this collection of research papers to Miklós Csörgő as a token of respect and appreciation of his work in Probability and Statistics by all the contributors to this volume, and all the participants of ICAMS'02. We are grateful to the contributors for submitting their papers for publication in this volume, as well as to the referees for their valuable time and enhancing work on it. All papers have been refereed, and accordingly revised if so requested by the editors. We wish to record here our sincere thanks to everyone for their further time, care and collaboration throughout this elaborate process.

The papers in this volume reflect the wide ranging interest of Miklós in Probability and Statistics, and nearly all of them are connected to his research. The editors also have a 69 page résumé of his work over the past forty or so years, titled Path Properties of Forty Years of Research in Probability and Statistics: In Conversation with Miklós Csörgő. This article, together with Miklós's list of publications is available as No. 400–2004 of the Technical Report Series of LRSP. It can also be accessed on the LRSP website: `www.lrsp.carleton.ca`, as well as on the Fields Institute website: `www.fields.utoronto.ca/publications/supplements/`. Unfortunately, due to space limitations, we could not include this résumé with its 311 references and Miklós's list of publications in this collection.

More than half of the 28 papers in this volume are up-to-date surveys on various active research areas in Probability and Statistics. All the sections except for **Part 2** are headed by survey papers that are also indicative of the main themes of these sections. All other papers, including **Part 2** in which there are three survey papers on different themes, are alphabetically ordered.

In **Part 1 Csáki, Földes** and **Shi** provide a survey of their joint work with Miklós on path properties of stochastic processes, a most insightful review of their collaboration on strong approximations of local time and additive functionals, path properties of Cauchy principal values of Brownian local time, iterated processes, level crossings of the empirical process, Vervaat and Vervaat-error processes and Banach space valued stochastic processes. **Khoshnevisan** presents a self-contained theory of quasi-sure results via Brownian sheet connections. In the first of their two papers in this volume **Peccati** and **Yor** provide a unified framework by means of Hardy's inequality in $L^2[0,1]$ for two results concerning the existence of certain integrals associated with a one dimensional Brownian motion starting from zero and the principal values of Brownian local times. In the second of their two papers **Peccati** and **Yor** generalize and give new proofs of four limit theorems on quadratic functionals of Brownian motion and Brownian bridge that were recently obtained by Deheuvels and Martynov, and establish explicit connections with occupation times of Bessel processes, Poincaré's Lemma and the class of quadratic functionals of Brownian local times studied in their preceding paper. Estimating the local time of a Wiener process from its values at integers, **Révész** provides a new look at one of his results of twenty years ago with Miklós.

The papers in **Part 2** survey several new directions in probability theory and its applications. **Bhansali, Holland** and **Kokoszka** study properties of chaotic maps that provide non-linear, non-Gaussian models as alternatives to earlier established linear and Gaussian stochastic models for the class of discrete-time long-memory stationary processes. **Davydov** and **Paulauskas** survey recent results on, and give a short introduction to, p–stable convex compact sets in Banach spaces, with special attention to stable random zonotopes. **Davydov** and **Zitikis** survey results on convex rearrangements, called by them convexifications, of stochastic processes. They also provide a view of relationships of convexifications with the operators of monotone and convex rearrangements in functional analysis and with the generalized Lorenz curves of econometrics. **Dawson, Gorostiza** and **Wakolbinger** survey recent work on hierarchical random walks with emphasis on transience-recurrence phenomena and in particular on the notion of degree of transience. They also describe a family of hierarchical random fields. Applications of hierarchical random walks and fields in statistical physics and branching processes are also discussed. Studying the expected distance from the origin after n steps of the so-called isotropic Pearson random walk in the plane, **Ross** and **Shao** improve the upper bound on Helgason's number.

The classical Erdős–Hsu–Robbins notion of complete convergence has led to various extensions of this original idea. One of these extensions is Heyde's notion of what is now called precise asymptotics. Following their extensive survey of precise asymptotics for sums, in **Part 3 Gut** and **Steinebach** extend these results to renewal counting processes and first passage time processes of random walks. As an analog of Heyde's theorem for ordinary means, Sándor **Csörgő** obtains the precise asymptotic behaviour of bootstrap means.

In **Part 4, Ćwiklińska** and **Rychlik** present necessary and sufficient conditions for the weak convergence of random sums and maximum random sums of independent random variables. **Tomkins** concludes necessary and sufficient conditions for the almost sure and complete stability of weighted maxima of bounded i.i.d. random variables.

Part 5 is devoted to change-point analysis. **Hušková** contributes a survey of procedures based on permutation tests or resampling methods for obtaining approximations to the critical values of various test procedures for detecting changes in statistical models. **Aly** proposes and studies L–statistics based test procedures for detecting a change in the distribution of a random sample. **Atenafu** and **Gombay** define truncated sequential tests via the generalized likelihood ratio to detect change in observations described by the nested random effects model. **Orasch** studies the asymptotic behaviour of U–statistics based processes whose appropriate functionals can be used to detect multiple changes in the distribution of a sample of possibly vector valued observations.

In **Part 6**, based on quantiles, comparison distributions, and conditional quantiles, **Parzen** develops a unified non-parametric framework which he calls "Statistical methods learning: for understanding and applying statistical methods." Using a strong martingale approach to weak convergence, **Burke** considers cumulative sum processes that can be used to test the fit of models in multivariate regression and the proportional hazards model of survival analysis. Given certain marginals, **Dabrowski** and **Dehling** study the conditional distribution of a multinomial sample and obtain a local multivariate normal limit theorem. As a consequence they prove asymptotic normality of the so-called H–coefficient in certain nonparametric unfolding models with dichotomous data. **Ghoudi** and **Rémillard**, continuing their work that was published in the above mentioned volume [1], provide a unified treatment of inference procedures that are based on pseudo-observations in the multivariate setting, and give several examples of applications as well.

Part 7, devoted to applications to economics, opens with a review of recent advances in the probabilistic and statistical theory of GARCH and related processes by **Berkes, Horváth** and **Kokoszka**. GARCH type models are extensively used in modeling returns on speculative assets. **Kulperger's** paper deals with aspects of contingent claim pricing in the incomplete discrete time model for returns via seeking a method to choose amongst members of the family of risk neutral measures that is close in some sense to the historical model measure. **McLeish** builds on using high and low price records of financial time series for estimating volatility parameters and correlation, and finds a multivariate normal approximation to the joint distributions of high, low and close price records to be a useful tool for pricing certain path-dependent options. **Yu** in his paper surveys recent developments on asymptotic results for residual processes of (G)ARCH time series models, and shows that, though most common processes such as partial sums and empirical processes have Gaussian limits that depend on the unknown parameters of these models, some of these processes when properly normalized will have a Gaussian limit that is free of model parameters. Hence one can, for example, test for model fitness or model misspecification in such situations.

In **Part 8, Csörgő, Szyszkowicz** and **Wang** survey weighted approximations in probability and strong limit theorems for self-normalized partial sums processes. In the last paragraph of our above mentioned résumé in conversation with Miklós Csörgő (cf. [2]), we mention a number of further important references on invariance principles in this regard. In the second paper of **Part 8, Wang** sharpens an earlier result on a Darling-Erdős type theorem for self-normalized sums.

We now wish to take this opportunity to sincerely thank the Natural Sciences and Engineering Research Council (NSERC) of Canada for their financial support of our LRSP by their Major Facility Access (MFA) awards in these years. Without these MFA awards, it would have been impossible for our LRSP to exist, and for us to even think about organizing our ICAMS'02. We are also grateful to the Fields Institute for Research in Mathematical Sciences for their financial support of our conference. We hope very much that this volume and the international success of ICAMS'02 will have contributed to the justification of their trust in us.

Last, but not least, we most sincerely wish to thank Gillian Murray, the Coordinator of our manifold LRSP activities, for her help throughout, for taking care of all the logistics of ICAMS'02, and for her invaluable technical skill and role in preparing this volume for publication.

In conclusion, we also want to express our appreciation to the Editorial Board of the Fields Institute for their approval of the publication of these proceedings in their Communications Series, to Carl R. Riehm, the Managing Editor of Publications, and Tom Salisbury, the Deputy Director, for their kind attention to, and sincere interest in, the publication of this volume, and to their Publications Manager, Debbie Iscoe, for her cooperation and expert help in its preparation for the AMS publishers. We hope very much that the readers will find this collection of papers

informative and also helpful in their studies and work.

LAJOS HORVÁTH
BARBARA SZYSZKOWICZ

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Path Properties of Forty Years of Research in Probability and Statistics: In Conversation with Miklós Csörgő

1 The Sixties and the First Half of the Seventies With Occasional Glimpses Into Some of the Years After

After two years of qualifying studies in Mathematics at McGill, and many odd jobs in between in Montreal, in 1959 Miklós Csörgő (B.A., Economics, Budapest, 1955) was accepted as a graduate student in the Department of Mathematics of McGill University. At the same time, in support of his studies, he was offered a part-time job as programmer in McGill's first, then brand new (IBM 650), Computing Centre, which he was glad to accept. Miklós Csörgő arrived in Canada on January 16, 1957, where he became a landed immigrant together with many thousands of other refugees, who left Hungary after the defeat of the October 23, 1956 Hungarian revolution.

Empty cell tests. The just mentioned part-time job at the McGill Computing Centre had also led to the first publication, [**1**] with Irwin Guttman in 1962, in which, inspired by an early version of the S.S. Wilks paper [301], tables of the (approximate) 1% and 5% critical values are provided for the *one and two sample empty cell tests* with a new derivation of their probability function under the null hypothesis that the two samples have come from the same population. The tables themselves were tabulated on the above mentioned McGill IBM 650, using a Fortran program. Paper [**1**] led to [**2**] with Irwin Guttman in 1964, concerning the consistency of the two-sample empty cell test.

M.A. (1961)**, Ph.D.** (1963)**, Postdoctoral** (1963-65) A National Research Council of Canada Studentship Award in 1960 enabled Miklós Csörgő to continue his graduate studies at McGill for three more years, where he obtained his M.A. (McGill, 1961) and Ph.D. (McGill, 1963), both in Mathematics, under the inspiring, helpful and most encouraging supervision of W.A. O'N. Waugh. His M.A. thesis was titled "*Axioms for Conditional Probability Spaces*", an essay on Alfréd Rényi's fundamental work [225] of 1955 on the foundations of conditional probability spaces, and related measure theoretic matters. His Ph.D. thesis was titled "*Some Kolmogorov–Smirnov–Rényi Type Theorems of Probability*". It was motivated by Rényi's landmark paper [224] of 1953 on the theory of order statistics, where he also studied weighted versions of the classical Kolmogorov–Smirnov statistics. For illustration of this idea, let F be the continuous distribution function of a random variable X, and F_n the empirical distribution function of a random sample $X_1, \ldots, X_n$, taken on X. Rényi [224] introduced and studied the asymptotic behaviour of functionals like

$$\sup_{a \le F(x) \le 1} (F_n(x) - F(x))/F(x) \quad \text{and} \quad \sup_{0 \le F(x) \le b} (F_n(x) - F(x))/(1 - F(x)),$$

as well as that of their two-sided versions. His idea of introducing these modifications of the classical Kolmogorov–Smirnov statistics was to make them more sensitive to detecting deviations on the tails from a hypothesized distribution F.

Two most inspiring *postdoctoral years* followed, 1963–65, in the Department of Mathematics of Princeton University as Instructor, and Postdoctoral Fellow on research funds of William Feller and John W. Tukey. Doing further work along the lines of his just mentioned 1963 Ph.D. thesis resulted in the publications [**3**]–[**10**] during 1965–67. Some of these works were discussed in Rényi's paper [229] in 1969, and commented on as well in Endre Csáki's Candidatus of Sciences dissertation [60] in 1974 (cf. also [63]). The latter provide a unified treatment for one-sided Kolmogorov–Smirnov–Rényi type problems concerning empirical distributions via combinatorial methods, variations on the so-called ballot lemma *à la* Lajos Takács [282] and H.E. Daniels [84].

Random sums, Rényi-mixing, Poissonization. Paper [**11**] is a first excursion into *sums* and *random sums* of *absolutely fair random variables* and their *Rényi-mixing* (cf. [226] and [230]). This in turn led to collaboration with Roger Fischler and Sándor Csörgő on related topics, as in [**13**], [**20**] and [**23**] and [**14**], [**24**] respectively. For further results along these lines we refer to [**15**] with Mayer Alvo, and to [**21**], [**26**]. One of the earliest papers dealing with random limit theorems was published by Mark Kac [159] in 1949. He introduced *Poisson random-size samples* for the sake of studying the problem of the asymptotic distribution of the uniform empirical process. With proving rigorously that the sup-functional of the absolute value of his Poisson random-indexed uniform empirical process converges in distribution to that of a Wiener process, Kac [159] came pretty close to proving also that the sup-functional of the absolute value of the uniform empirical process must converge in distribution to that of a Brownian bridge, the very question J.L. Doob was posing in his famous "Heuristic approach..." paper [104] in the same year. Kac's idea of *Poissonization* turned out to be very useful also later on, when the need for establishing strong invariance principles for *multivariate empirical processes* became apparent after Kiefer's landmark paper [175] in 1972 (cf. e.g., M.J. Wichura [300], [**28**] and [**32**] with Pál Révész, Révész [235], [**101**] with Lajos Horváth). The above mentioned seminal work of Kac and the many further papers which also make use of the idea of Poissonization and, indeed, the very *Skorohod* [267] *embedding scheme* itself, show that the notion of randomly stopped processes has played a significant role in the development of our view of the *invariance principle* that was first conceived and used by Pál Erdős and Mark Kac in their celebrated ground breaking paper [116]. On the other hand, the exposition [**31**], in Hungarian, with S. Csörgő, R. Fischler and P. Révész shows how some fundamental advances in the theory of strong invariance principles in the mid-seventies can, in turn, be applied to studying similar strong and weak *convergence properties of randomly selected sequences*, such as *empirical* and *quantile processes when the sample size is random*, or *partial sums of a random number of random variables*. For a review of this paper we refer to **MR58 #13249** by Endre Csáki. We note as well that Chapter 7 of the by now classic book [A1] with Pál Révész is based on the paper [**31**] that also lists a bibliography of 69 related papers. For further references along these lines we refer to J.-E. Karlsson and D. Szász [162].

Replacing composite goodness-of-fit hypotheses by equivalent simple ones. Typical goodness-of-fit problems are concerned with testing for a random sample possibly coming from a specific probability distribution of statistical interest. If the distribution in hand is not completely specified by the so-called null assumption about it, then we have a composite goodness-of-fit hypothesis to deal with. Starting with paper [**12**] with V. Seshadri and M.A. Stephens, and continuing with [**16**], [**18**], [**19**], [**22**], [**25**], [**30**], these papers deal with the problem of *replacing composite goodness-of-fit hypotheses by equivalent simple ones.*

Strongly multiplicative systems. A sequence $\xi_1, \xi_2, \ldots$ of random variables is called an *equinormed strongly multiplicative system* (ESMS) (cf. G. Alexits [7]) if

$$E\xi_i = 0, \quad E\xi_i^2 = 1, \quad i = 1, 2, \ldots, \tag{1.1}$$

and

$$E\left(\xi_{i_1}^{r_1}\xi_{i_2}^{r_2}\cdots\xi_{i_2}^{r_k}\right) = E\left(\xi_{i_1}^{r_1}\right)E\left(\xi_{i_2}^{r_2}\right)\ldots E\left(\xi_{i_k}^{r_k}\right), \quad k = 1, 2, \ldots, \tag{1.2}$$

where $r_1, r_2, \ldots, r_k$ can be equal to 1 or 2. If (1.2) holds for $k = 2, \ldots, K$, then this system of random variables is called a *K–wise* ESMS.

G. Alexits [7], G. Alexits and K. Tandori [8] showed (cf. Theorem 3.3.1 in Révész [234]) that a *uniformly bounded* ESMS satisfies

$$\sum_{k=1}^{\infty} c_k \xi_k < \infty \qquad \text{with probability 1} \tag{1.3}$$

with a sequence of real numbers $c_1, c_2, \ldots$ for which

$$\sum_{k=1}^{\infty} c_k^2 < \infty. \tag{1.4}$$

We note in passing that the condition (1.4) implies (1.3) for a mean zero sequence of independent random variables, and that the latter turned out to be also true for orthogonal sequences as well in the special case of Fourier series (cf. L. Carleson [42]). Orthonormal systems of mean zero random variables $\xi_1, \xi_2, \ldots$ in general require the stronger condition

$$\sum_{k=1}^{\infty} c_k^2 \log^2_{\text{base2}} k < \infty \tag{1.5}$$

for having (1.3) (cf. Theorem 3.2.1 in Révész [234]). Tandori [284] showed that, if $c_1, c_2, \ldots$ is a monotonically decreasing sequence of real numbers for which the series (1.4) diverges, then there exists an orthonormal system $\xi_1, \xi_2, \ldots$ such that the series $\sum_{k=1}^{\infty} c_k \xi_k$ is nowhere convergent (cf. Theorem 3.2.4 in Révész [234]). Thus, in general, for an orthonormal system of random variables to behave as in (1.3), the sufficient condition of (1.4) is, essentially, also necessary. In view of this result did Alexits [7] and Alexits and Tandori [8] propose the study of various *multiplicative* systems, aiming at finding conditions that would imply the almost everywhere convergence of the series $\sum_{k=1}^{\infty} c_k \xi_k$, where $\xi_1, \xi_2, \ldots$ is an orthonormal system of random variables and $\sum_{k=1}^{\infty} c_k^2 < \infty$. Moreover, they showed that such results were feasible for what they called an ESMS as above (cf. (1.1) and (1.2)), provided the latter is assumed to be *uniformly bounded* (cf. (1.3) via (1.4)). Révész proved an analogue of this result for 4-*wise* uniformly bounded ESMS (cf. Theorem 3.3.4 of [234]), and in [233] he proved the *first* LIL (cf. Theorem 3.3.3 of [234]) and CLT for a uniformly bounded ESMS of random variables.

Inspired by Révész's first LIL in [233] for an ESMS of uniformly bounded random variables and by the rest of his papers along these lines later on (for a review of various multiplicative systems of random variables and further references, we refer to [**186**]), in [**17**] Csörgő studied the LIL problem for normed strongly multiplicative systems (NSMS) of random variables $\xi_1, \xi_2, \ldots$ that satisfy (1.1) with $E\xi_i^2 = \delta_i^2$, $i = 1, 2, \ldots$, as well as the product rule of (1.2). For such an NSMS of random variables that are *not* assumed to be *uniformly bounded*, a LIL was proved in [**17**] that, in the light of conditional medians as in R.J. Tomkins [287], was extended in [**35**] with Don McLeish to read as follows: *Let* $\{\xi_i\}_{i \geq 1}$ *be an* NSMS *as just defined, put* $S_k = \xi_1 + \cdots + \xi_k$, $\hat{\delta}_n^2 = \delta_1^2 + \cdots + \delta_n^2$ *and assume that, as* $n \to \infty$,

$$\hat{\delta}_n^2 \to \infty, \quad |\xi_n| / \hat{\delta}_n = o((\log \log \hat{\delta}_n^2)^{-1/2}),$$

and that we have

$$\sum_k^n /\hat{\delta}_n^2 \to 1 \quad a.s., \textit{ for every } \; k \geq 1,$$

where $\sum_k^n := E\Big((S_k - S_n)^2 | S_1, \ldots, S_k\Big)$. *Then, for every* $k \geq 1$,

$$\limsup_{n\to\infty} |S_n|/(2\Sigma_k^n \log\log \Sigma_k^n)^{1/2} \leq 1 \quad a.s. \tag{1.6}$$

The constant 1 in (1.6) is likely best possible. If the latter were to be true, even under some additional conditions, the just quoted result could be viewed as an NSMS version of the W.F. Stout [271] martingale analogue of Kolmogorov's LIL.

Randomly indexed sequences revisited in the milieu of the subsequence principle. Another way of aiming at results like having (1.3) via (1.4) for a sequence of random variables that are bounded in some sense, moments-wise or otherwise, is that of the *subsequence principle*. Initiated by H. Steinhaus (cf. *The New Scottish Book* (Wroclaw, 1946–1958), Problem 126), this notion has evolved via realizing (cf. Révész [232], Komlós [180], Chatterji [45], [46], [47]) that the phenomenon of subsequences of special orthogonal sequences, like for example lacunary *subsequences* of trigonometric and Walsh functions, *behaving like sequences of independent random variables* is not restricted only to such distinctive sequences. Inspired by [232] and [180], Chatterji, in a series of papers starting with [45] and exemplified by and listed in [46] and [47], continued to establish more general corresponding analogues of the classical properties of independent sequences, and propounded his *subsequence principle* (for more details we refer to [**186**]). This principle asserts that every limit property enjoyed by all independent identically distributed sequences satisfying some moment condition is shared by some, nonrandomly chosen, subsequence of every sequence which satisfies the moment condition uniformly. Thus it became a challenge (cf. J.F.C. Kingman [178]) to establish a general result to embrace all known special cases that had to be proved in each instance of the principle so far. This, in turn, was achieved by D.J. Aldous [2], [3], [4] via a general fundamental theorem subsuming all the known cases of the subsequence principle till that time. Moreover, the latter papers have also given a fresh impetus to the study of *randomly indexed sequences of random variables*, as well as to that of *Rényi mixing* and *stable limit theorems* (cf. D.J. Aldous [5], D.J. Aldous and G.K. Eagleson [6]).

The papers [**48**] and [**53**] with Zdzisław Rychlik were written in this renewed milieu for randomly indexed sequences of random variables. For a short glimpse, let (S, d) be a separable metric space equipped with its Borel σ–field $\mathcal{B}$. Let $\{Y_n, \, n \geq 1\}$ be a sequence of S–valued random elements defined on a probability space $(\Omega, \mathcal{A}, P)$, and let $\{N_n, \, n \geq 1\}$ be a sequence of positive integer-valued random variables defined on the same probability space. Assuming $Y_n \xrightarrow{\mathcal{D}} Y$ in (S, d) and $N_n \to \infty$ as $n \to \infty$, then what further hypotheses are needed to deduce $Y_{N_n} \xrightarrow{\mathcal{D}} Y$ in (S, d)? Much of the previous work on this problem has used *Anscombe's condition* (cf. [11]), i.e., his "uniform continuity" hypothesis: *for each* $\epsilon > 0$ *there exists* $\delta > 0$ such that

$$\limsup_{n\to\infty} P\left(\max_{|i-n|<\delta n} d(Y_i, Y_n) \geq \epsilon\right) \leq \epsilon, \tag{1.7}$$

in combination with $N_n/k_n \xrightarrow{\mathrm{P}} \lambda$, where λ is a positive random variable and k_n are constants going to infinity as $n \to \infty$. Aldous [5] proved that Anscombe's condition that is designed for applications to normalized partial sums of stationary sequences is exactly the right one when $N_n/k_n \xrightarrow{\mathrm{P}} 1$. He also gave some necessary and sufficient conditions for $Y_{N_n} \xrightarrow{\mathcal{D}} Y$ in the case when $N_n/k_n \xrightarrow{\mathrm{P}} \lambda$. It is shown in [**48**] that for applications to the nonstationary case a different hypothesis is appropriate, and analogues of known theorems are obtained accordingly. Roughly speaking, the *theorems in* [**48**] *extend* Aldous' *results in* [5] to the nonstationary case (e.g., independent summands satisfying Lindeberg's condition) *via introducing a new version of Anscombe's condition*, as well as a new assumption on the sequence of positive integer-valued random variables that index the sequence of random elements in hand.

Paper [**54**] with Rychlik introduces yet another version of Anscombe's condition which in terms of real valued $\{Y_n,\, n \geq 1\}$ reads as follows: *a sequence* $\{Y_n,\, n \geq 1\}$ *of random variables is said to satisfy the generalized Anscombe condition with norming sequences of numbers* $\{k_n,\, n \geq 1\}$ *and* $\{w_n,\, n \geq 1\}$ *if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\limsup_{n\to\infty} P\left(\max_{i\in D_n(\delta)} |Y_i - Y_n| \geq \epsilon w_n\right) \leq \epsilon, \tag{1.8}$$

where $D_n(\delta) := \{i : |k_i^2 - k_n^2| \leq \delta k_n^2\}$.

When $k_n^2 = n$, $n \geq 1$, the condition (1.7) reduces to that of (1.6), i.e., to Anscombe's condition in [11]. If $w_n = 1$, $n \geq 1$, then (1.6) reduces to the version used in [**48**].

In [**53**] it is shown that the following two conditions are equivalent. (i) $\{Y_n,\, n \geq 1\}$ *satisfies the generalized Anscombe condition* (1.7) *with norming sequences of positive numbers* $\{k_n,\, n \geq 1\}$ *and* $\{w_n,\, n \geq 1\}$*, and, as* $n \to \infty$*, the sequence* $(Y_n - \theta)/w_n$ *converges weakly to a probability measure* μ*, where* θ *is a real number.* (ii) *For every family* $\{N_t,\, t \geq 1\}$ *of positive integer-valued random variables, as* $t \to \infty$, $\{Y_{N_t} - \theta)/w_{a_t},\, t \geq 1\}$ *converges weakly to* μ*, provided* $k_{N_t}^2/k_{a_t}^2$ *tends to* 1 *in probability as* $t \to \infty$*, where* $\{a_t,\, t \geq 1\}$ *is a family of positive integers such that* $a_t \to \infty$ *as* $t \to \infty$. The implication that (ii) implies (i), in the special case $k_n^2 = n$, $n \geq 1$, gives an affirmative answer to Anscombe's conjecture in [11] on page 607. As a simple consequence of the equivalence of (i) and (ii), Theorem 4 of [**53**] provides a method for determining sequential stopping rules via giving a required accuracy of estimation of an unknown parameter. In particular, *Section 3* of [**53**] generalizes Anscombe's Theorem 2 in [11], and *gives a general procedure for estimating, with given small standard error, the mean of some population, as well as for finding a confidence interval of prescribed width and prescribed probability coverage for the unknown mean of a population.* Thus [**53**] extends some of the results that are presented in Anscombe [12], and Chow and Robbins [54]. Paper [**53**] is also concerned with *Rényi mixing* and *stable* randomly indexed limit theorems (cf. Rényi [223], [226], [227], [228], Rényi and Révész [230], Kátai and Mogyoródi [161], Eagleson [107], Fischler [127], [**23**] with Fischler, [**24**] with S. Csörgő, Aldous and Eagleson [6], Aldous [5], and for more recent references and results S. Csörgő [77] in [V1], Kowalski and Rychlik [184] in [V1], Ćwiklińska and Rychlik [82] in [V2]). In this regard Theorem 3 of [**53**] extends the results obtained by J.R. Blum, D.L. Hanson and J.I. Rosenblatt [29], and S. Guiaşu [137].

2 From Mid-seventies to Mid-eighties, a New Course of Events in Strong Invariance, Their Impact on Gaussian and Related Processes: An Interplay Then and Beyond

Invariance principles have evolved from two major sources: partial sum processes and empirical processes. The main papers that have led to the theory of weak convergence in metric spaces are Erdős and Kac [116], and Donsker [101] on partial sum processes, and Doob [104] and Donsker [102] on empirical processes. Prohorov [216] and Skorohod [266] gave the theory its present form, as it is fully explored, expanded and formalized in Billingsley [27]. A completely new point of view, namely the notion of strong invariance, was introduced in this subject by Strassen [272] with the help of the Skorohod [267] embedding scheme. His results have generated an activity which has played a crucial role in our understanding of what randomness is all about.

The first two joint papers of Miklós Csörgő with Pál Révész (cf. [**27**], [**28**]) were born in, and inspired by, this mathematical milieu. Changing only numberings and the style of mentioning references to fit the present volume, as well as our exposition, we quote from his tribute to Pál Révész [**186**] in [B3], incorporating his lines into our own presentation.

Approximating partial sums and empiricals. On and around the first two papers with Pál Révész, as described by Miklós Csörgő. "As we have just noted above, our first joint papers (cf. [**27**], [**28**]), respectively titled "A new method to prove Strassen type laws of invariance principle. I & II", *appeared* in 1975. The first one of them was *received* on August 3, 1973, while the second one on November 30, 1973, both by ZfW. In the first round both were quickly and summarily *rejected*. They have eventually been rescued for publication in the same journal by Jack Kiefer, and thus *the first steps* of the *'Hungarian construction school'* have also become official.

The background of, and our inspiration for, paper [**27**] were as follows. Using the Skorohod embedding scheme, V. Strassen [273] *showed* that independent identically distributed random variables $X_1, X_2, \ldots$, with $EX_1 = 0$, $EX_1^2 = 1$, $E|X_1|^4 < \infty$, can be constructed on the same probability space as a standard Wiener process (Brownian motion) $\{W(t),\, 0 \le t < \infty\}$ so that, as $n \to \infty$,

$$Z_n := |S(n) - W(n)| = O((n \log\log n)^{1/4} (\log n)^{1/2}) \quad \text{a.s.}, \tag{2.1}$$

where $S(n) := \sum_{i=1}^{n} X_i$, and throughout as well.

In the same paper in 1965 Strassen also *posed* the following question. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with mean zero and variance one, and let $\{W(t),\, 0 \le t < \infty\}$ be a standard Wiener process on the same probability space such that, as $n \to \infty$,

$$|S(n) - W(n)| = o((n \log\log n)^{1/4} (\log n)^{1/2}) \quad \text{a.s.} \tag{2.2}$$

Is it true then that the distribution of X_1 is standard normal?

J. Kiefer [173] proved that, *in case of using the Skorohod* [267] *embedding scheme* with stopping times, say $\{T_i,\, i \ge 1\}$, for the sake of establishing (2.1) under the there stipulated four moments conditions of Strassen [273], one has the following *exact version* of (2.1)

$$\limsup_{n\to\infty} |Z_n| / ((n \log\log n)^{1/4} (\log n)^{1/2}) = (2\beta)^{1/4} \quad \text{a.s.} \tag{2.3}$$

where $\beta := \mathrm{var}(T_1)$. Consequently, when $\beta = 0$, then $X_1 \stackrel{\mathcal{D}}{=} N(0,1)$, a standard normal random variable. Thus, *in this case, Kiefer answered Strassen's question affirmatively* via concluding that replacing O by o in (2.1) does indeed imply that X_1 is a standard normal random variable. In other words, the rate of convergence in (2.1) cannot be improved via the Skorohod embedding scheme, no matter what further restrictions one might put on the distribution function of the random variable X_1.

The answer to Strassen's question in general, i.e., to posing the problem as stated in (2.2), remained open. Moreover, in the light of Kiefer's just quoted answer, any improvement of the rate of convergence of the strong approximation in (2.1) could only come from a totally *different* method of construction. A *new method of construction*, the so-called *quantile transform method*, was first developed in P. Bártfai [15] (cf. discussion of (2.5) and (2.6) below) and paper [**27**], in the latter for the sake of *giving a negative answer to Strassen's general question* as it is posed via (2.2) above. Namely, roughly speaking, in [**27**]. (cf. also Theorem 2.5.1 and Section 2.6 of book [A1]) we established that, on assuming Cramér's condition for the characteristic function of the random variable X_1, for any given $0 < \epsilon < 1/2$ one can assume enough moment conditions for X_1 of the independent identically distributed sequence $X_1, X_2, \dots$ with $EX_1 = 0$, $EX_1^2 = 1$, $EX_1^3 = 0, \dots$ such that with an appropriately constructed standard Wiener process $\{W(t),\, 0 \le t < \infty\}$ on the same probability space, as $n \to \infty$, we have

$$|S(n) - W(n)| = o(n^{\epsilon}) \quad \text{a.s.} \tag{2.4}$$

Following this line of thought, J. Komlós, P. Major, G. Tusnády [KMT] [181], [182] and P. Major [191] ingeniously refined our quantile transform method and proved that, on assuming only that the mean zero and variance one random variable X_1 *has $p > 2$ moments,* (2.4) *holds true with $\epsilon = 1/p$*, and that this is the *best possible* strong invariance principle under this assumption. As to X_1 *having only two moments*, P. Major [192] proved that, in this case the *first strong invariance principle*, namely that of Strassen [272], *is best possible.* Concerning the question of *Donsker's theorem* (cf. M. Donsker [101]) *via strong approximations*, we refer to P. Major, *Ann. Probab.* **7** (1979), 55–61, and to pages 112 & 113 of book [A1].

These questions and answers are connected with the so-called *"stochastic geyser problem"* (cf. Sections 2.2–2.4 of book [A1]). Suppose we assume that *we observe* the sequence

$$V_n = S(n) + R_n, \quad n = 1, 2, \dots, \tag{2.5}$$

where $\{R_n\}$ is an arbitrary sequence of random variables. In statistical terminology the latter can be viewed as a random error sequence when trying to observe $\{S(n)\}$ in order to estimate the distribution function of the random variable X_1. A theorem of P. Bártfai [15] states that if $R_n = o(\log n)$ a.s., and X_1 has a finite moment generating function, then the sequence $\{S(n) + R_n\}$ determines the distribution function of X_1 with probability one. Now, reformulating this result in a strong invariance context via putting $R_n = S(n) - W(n)$, one concludes (cf. Theorem 2.3.2 of book [A1]) that X_1 must be a standard normal random variable. This formulation of Bártfai's just mentioned result of 1966 as a *lower limit to the strong invariance principle for partial sums* of random variables was however realized only later on by KMT [181], [182], who thus also concluded that the *best possible strong approximation for any versions of $S(n)$ and $W(n)$* should be

$$|S(n) - W(n)| = O(\log n) \quad \text{a.s.}, \tag{2.6}$$

and proved as well that *the latter* held true for all those possible distributions for X_1 which have a moment generating function in a neighbourhood of zero.

For a review of these and further related results we refer to Chapter 2 of book [A1], Chapter 1 of M. Csörgő and L. Horváth [A4] and, for weighted approximations of partial sum processes, to [**105**] and B. Szyszkowicz [278], [280]. For an *extension* and refinement of the KMT results *to the non–i.i.d. case of partial sums*, we refer to A.I. Sakhanenko [245], [246], [247], E. Berger [19], U. Einmahl [110], [111] and Qi-Man Shao [256]. For a *review of various approaches to proving invariance principles* we refer to W. Philipp [215]. For an *extension* of the Hungarian construction approach *to sums of vector valued random variables* we refer to U. Einmahl [110], [111], [112] and, for a glimpse at some more recent developments in this regard, to A. Yu. Zaitsev

[310] and the references therein. A very useful *companion work* on approximating partial sums of vector valued random variables is the I. Berkes and W. Philipp [22] paper. For example, by combining the Berkes-Philipp blocking technique and the CsR quantile transform methods of papers [**27**] and [**28**], Hao Yu [308] succeeds in almost surely approximating *partial sums* of an *associated sequence* of random variables by appropriate partial sums of another associated sequence with Gaussian marginals.

Though we have completed and submitted paper [**27**] to ZfW a bit earlier than [**28**], we were initially intrigued and inspired by the landmark paper of J. Kiefer [175] and initiated our work in conjunction on both. Our driving force for writing [**28**] was to understand Kiefer's fundamental results in the latter paper, in which he was *first to construct an almost sure representation of the empirical process by an appropriate two-time parameter Gaussian process.* In particular, J. Kiefer [175] succeeded in embedding the empirical process into an appropriate two-time parameter Gaussian process via his ingenious extension of the Skorohod [267] embedding scheme to the case of vector valued random variables. Due to our great admiration for these landmark achievements, in paper [**28**] we called the latter Gaussian process *the Kiefer process.* D.W. Müller [202] gave a proof of the *convergence* in law of the empirical process to a two-time parameter Gaussian process of the same appropriate covariance function, as well as the first estimate of the error for the convergence in distribution of certain functionals of the *sequence* of empirical processes.

Let $U_1, \ldots, U_n$ $(n = 1, 2, \ldots)$ be independent random d-vectors, uniformly distributed on $I^d := [0,1]^d$, $d \geq 1$, and for each n define the *uniform empirical process* of these random variables by

$$\alpha_n(y) := \sqrt{n}\left(\frac{\sum_{i=1}^n \mathbb{1}_{[0,y]}(U_i)}{n} - \lambda(y)\right), \quad y \in I^d, \tag{2.7}$$

where $\lambda(y)$ stands for Lebesgue measure of the d–dimensional interval $(0, y]$.

A separable mean zero *Gaussian process* defined on $[0,1]^d \times [0,\infty)$, $\{K(y,t);\, y \in I^d,\ t \in \mathrm{R}_+^1\}$, *is* called a *Kiefer process* if

$$EK(x,s)K(y,t) = (\lambda(x \wedge y) - \lambda(x)\lambda(y))(s \wedge t), \tag{2.8}$$

where the minimum $x \wedge y$ is meant to be taken component wise and $\lambda(\cdot)$ is Lebesgue measure as in (2.7), i.e., the Gaussian $K(\cdot,\cdot)$ is a Brownian bridge in its first argument and a Wiener process in its second argument. Note that $\sqrt{n}\alpha_n(y)$ as a two-time parameter process (non-Gaussian) has the same covariance structure as that of a Kiefer process (Gaussian).

Using now this terminology, the almost sure invariance principle of J. Kiefer [175] reads as follows: *One can construct a probability space for $U_1, U_2, \ldots$ with a Kiefer process $K(\cdot,\cdot)$ on it so that, as $n \to \infty$,*

$$\max_{1 \leq k \leq n} \sup_{y \in I^1} \left|\sqrt{k}\ \alpha_k(y) - K(y,k)\right| = O(n^{1/3}(\log n)^{2/3}) \quad \text{a.s.} \tag{2.9}$$

In paper [**28**] we *combine the classical Poissonization* technique (cf. M. Kac [159], and Chapter 7 of book [A1] for historical remarks) *with* our *quantile transform method* and establish the following result: *One can construct a probability space for $U_1, U_2, \ldots$ with a sequence of Brownian bridges $\{B_n(y);\, y \in I^d,\ d \geq 1\}_{n=1}^\infty$ on it so that for any $r > 0$ there is a constant $C > 0$ such that for each $n = 2, 3, \ldots$*

$$P\left\{\sup_{y \in I^d} |\alpha_n(y) - B_n(y)| > C(\log n)^{3/2} n^{-1/2(d+1)}\right\} \leq n^{-r}, \quad d \geq 1, \tag{2.10}$$

and with a Kiefer process $\{K(y,t); (y,t) \in I^d \times \mathbb{R}^1_+\}$ *on it so that for any* $r > 0$ *there is a constant* $C > 0$ *such that for all* $n \geq 2$

$$P\left\{ \max_{1\le k\le n} \sup_{y\in I^d} |\sqrt{k}\ \alpha_k(y) - K(y,k)| > Cn^{(d+1)/2(d+2)}(\log n)^2 \right\} \le n^{-r},\ d \ge 1. \tag{2.11}$$

These *"coupling processes"* type inequalities for the uniform empirical and their approximating Gaussian processes *are first of their kind* in the literature. They imply strong (almost sure) bridge type (resp., strong Kiefer type) invariance principles for $\alpha_n(y)$ with the rates $O(b_n)$ a.s. (resp., $O(k_n)$ a.s.), uniformly over the quadrants in I^d, $d \geq 1$, with

$$b_n = (\log n)^{3/2} n^{-1/2(d+1)} \text{ (resp. } \ k_n = (\log n)^2 n^{-1/2(d+2)}). \tag{2.12}$$

We note that with $d = 1$, the respective rates b_n and k_n in (2.12) are not as good as the respective corresponding ones of D. Brillinger [37] (cf. Theorem 4.3.1 in book [A1]) and J. Kiefer [175] (cf. (2.9) above) are, which were obtained via Skorohod schemes of embedding. They were however brand new for $d \geq 2$ at that time, and their quantile transform method of proof combined with the Poissonization technique was also new. Moreover, in the latter case of $d \geq 2$, this kind of combined approach to establishing better rates of the form as in (2.12) turned out to be also the right one (cf. P. Massart [200]).

Right after papers [**27**] and [**28**], the *dyadic scheme refinement of the quantile transform method* of KMT [181] resulted in their well known trade-mark exponential rates versions of (2.10) and (2.11) when $d = 1$, which in turn respectively yielded the best possible rate $b_n = (\log n)n^{-1/2}$ for the bridge type strong invariance principle, as well as the rate $k_n = (\log n)^2 n^{-1/2}$ for the strong Kiefer invariance principle when $d = 1$.

As to these first three landmark papers, i.e., [**27**] and [**28**] by CsR, and the just mentioned KMT [181] exposition, we also like to refer to MR**51** ##116005a,b, where in his foresighted review of them more than 25 years ago, Jack Kiefer wrote:

> These papers develop a new tool for obtaining error bounds in asymptotic theory, perhaps the most novel technique since Skorohod embedding, and one that improves strikingly on the latter. ... Extensions and refinements of all these results have since been announced by various subsets of these five authors. This approach thus offers great promise. ... The present stronger sample-space linkage has already exhibited its applicability to a wide variety of new problems.

Some 17 or so years later, in his **Foreword** to [A4] with Lajos Horváth, commenting on that book and the *Hungarian construction*, and counting [**27**] and [**28**] as one paper, David Kendall writes:

> The astonishing 'Hungarian construction' has become well known since its introduction in 1974-5 in two ZfW papers, one by Miklós Csörgő and Pál Révész and the other by Komlós, Major and Tusnády. The literature revolving around this important work is already immense. An account of it in book form was provided by Miklós Csörgő and Pál Révész in 1981, but that book in itself will have stimulated a demand for more, and now we have the latest (but surely not the last!) word on the subject, this time by Miklós Csörgő and Lajos Horváth.

> Obviously, the 'Hungarian construction' is here to stay, and it has already found applications to a wide variety of fields (including one to the archaeology of the Neolithic period). How delighted Alfréd Rényi would have been to see that happen!

Still in the seventies, in addition to [**27**], [**28**], the papers [**29**] [**32**] [**36**] [**40**] [235], [236], [40] and [**45**] have also played a fundamental role in the development and impact of the Hungarian construction school, and some of that we will now briefly mention in the context of skipping through some further advancements.

Concerning the uniform empirical process on $[0,1]$, which is easily extended to any arbitrary distribution on R^1, J. Bretagnolle and P. Massart [36] (cf. also Section 3.1 of [A4]) give a new proof for the KMT [181] approximation by Brownian bridges, while N. Castelle and F. Laurent-Bonvalot [43] reprove their approximation by a Kiefer process. Both papers also provide explicit constants in the respective KMT [181] inequalities.

Combining the classical Poissonization technique with the one dimensional *dyadic scheme refinement of the quantile transform method* of KMT [181], P. Massart [200], *in the context of Lebesgue measure* on $I^d = [0,1]^d$ as in (2.7) with $d \geq 2$, gets as good rates of approximation as possible in this particular multidimensional setting, both for Brownian bridge and Kiefer type approximations. For example, over the class of quadrants or the class of Euclidean balls in R^d, the Brownian bridge type strong invariance principle (resp. strong Kiefer type invariance principle) holds with $b_n = (\log n)^{3/2} n^{-1/2d}$ (resp. with $k_n = (\log n)^2 n^{-1/2(d+1)}$). Thus, over the class of quadrants in R^d, $d \geq 2$, these results improve on the respective rates of the first two strong invariance principles in this regard in paper [**28**] (cf. the respective quoted rates in (2.12) above). For the empirical distribution over the unit cube $I^2 = [0,1]^2$ the best available Brownian bridge type approximation is due to G. Tusnády [289] with the rate $b_n = (\log n)^2 n^{-1/2}$ that coincides with the KMT [181] strong Kiefer invariance principle for $d = 1$ (cf. also N. Castelle and F. Laurent-Bonvalot [43]). Lower bounds when $d \geq 2$ are due to J. Beck [17]. The results of P. Massart [200] also generalize Révész's fundamental work in [235] and [236] on strong invariance principles indexed by classes of sets with smooth boundaries, and improve on it as well when $d \leq 6$. For related works along these lines we refer to [**32**] and to R.M. Dudley and W. Philipp [105].

I.S. Borisov ([33], [34]) also uses KMT type constructions to prove strong invariance principles for empirical processes of multivariate random variables. The rates he obtains are less efficient than the b_n rates of P. Massart [200], but they hold for more general distributions than Lebesgue measure on $I^d = [0,1]^d$, $d \geq 2$. For example, in Borisov's case the rate $b_n = (\log n) n^{-1/2(2d-1)}$ for *approximating by Brownian bridges* is valid *for all distributions* over the class of all quadrants in R^d. Using the latter result of Borisov in combination with the method of paper [**28**] for constructing a strong Kiefer type invariance principle from a Brownian bridge type invariance principle, M. Csörgő and L. Horváth [**101**] obtain a *strong Kiefer invariance principle* that is valid *for all distribution functions* over all quadrants in R^d with the rate $k_n = (\log n)^{3/2} n^{-1/4d}$. The latter rate is best available in this context."

From empiricals to quantiles. Paper [**29**], the third paper with Pál Révész in 1975, initiates the study of the *uniform quantile process* in view of KMT [181] and Kiefer [174]. Let $U_1, U_2, \ldots, U_n$ $(n = 1, 2, \ldots)$ be independent copies of a random variable U, uniformly distributed over the interval $[0,1]$. Let

$$E_n(y) := (1/n) \sum_{i=1}^{n} \mathbb{1}_{(0,y]}(U_i), \quad 0 \leq y \leq 1,$$

denote the empirical distribution function based on $U_1, U_2, \ldots, U_n$, where $\mathbb{1}_A$ is the indicator function of the set A. Let $G_n = E_n^{-1}$ be the left-continuous inverse of E_n and, *à la* (2.7), define the *uniform empirical and quantile processes* over the interval $[0,1]$ respectively by

$$\alpha_n(y) := n^{1/2}(E_n(y) - y), \qquad 0 \le y \le 1, \tag{2.13}$$

$$u_n(y) := n^{1/2}(G_n(y) - y), \qquad 0 \le y \le 1. \tag{2.14}$$

We already mentioned the KMT [181] trade-mark exponential rates version of (2.10) and (2.11) when $d = 1$. Their respective statements read as follows: *One can construct a probability space for* $U_1, U_2, \ldots$ *with a sequence of Brownian bridges* $\{B_n(y);\, 0 \le y \le 1\}$ *on it so that one can define positive absolute constants* A, B, C *such that*

$$P\left\{ \sup_{0 \le y \le 1} |\alpha_n(y) - B_n(y)| > n^{-1/2}(x + A \log n) \right\} \le B e^{-Cx} \tag{2.15}$$

for all $x > 0$ *and integer* $n \ge 1$*, and one can also construct a probability space for* $U_1, U_2, \ldots$ *with a Kiefer process* $\{K(y,t);\, 0 \le y \le 1,\, t > 0\}$ *on it so that one can define positive absolute constants* A, B, C *such that*

$$P\left\{ \max_{1 \le k \le n} \sup_{0 \le y \le 1} |k^{1/2}\alpha_k(y) - K(y,k)| > (x + A \log n) \log n \right\} < B e^{-Cx} \tag{2.16}$$

for all $x > 0$ *and integer* $n \ge 1$.

The already mentioned new proof of the KMT inequality (2.15) by Bretagnolle and Massart [36] concludes it with $A = 12$, $B = 2$, and $C = 1/6$. For details on the Bretagnolle and Massart proof we also refer to Section 3.1 of book [A4].

The detailed proof of the KMT inequality (2.16) by Castelle and Laurent-Bonvalot [43] yields it with $A = 76$, $B = 2.028$, and $C = 1/41$.

Using the notation and language of description that we introduced right after (2.10) and (2.11), from (2.15) we conclude the rate $b_n = (\log n) n^{-1/2}$ for the *bridge type strong invariance* for α_n when $d = 1$, i.e.,

$$\sup_{0 \le y \le 1} |\alpha_n(y) - B_n(y)| = O((\log n) n^{-1/2}) \quad \text{a.s.}, \tag{2.17}$$

while (2.16) yields the rate $k_n = (\log n)^2 n^{-1/2}$ for the *strong Kiefer type invariance* principle for α_n when $d = 1$, i.e.,

$$\sup_{0 \le y \le 1} |\alpha_n(y) - n^{-1/2} K(y,n)| = O(n^{-1/2} (\log n)^2) \quad \text{a.s.}, \tag{2.18}$$

The *rate* of convergence *in* (2.17) *is best possible* (cf. KMT [181]), while that in (2.18) cannot be improved beyond $(\log n)$ (cf. also Section 4.4 of [A1] and Theorem 3.1.2 of [A4]).

For further reference we quote also one of the KMT ([181], [182]) strong approximations for partial sums $S_n := \sum_{i=1}^n X_i$, $n \ge 1$, $S_0 := 0$, of i.i.d. random variables $X, X_1, X_2, \ldots$ with mean zero and variance one: *Assume that* $E \exp(tX) < \infty$ *in a neighbourhood of* $t = 0$*. Then one can construct a probability space for these random variables with a Wiener process* $\{W(t);\, 0 \le t < \infty\}$ *on it so that*

$$P\left\{ \sup_{0 \le t \le T} |S_{[t]} - W(t)| > A \log T + x \right\} \le B e^{-Cx} \tag{2.19}$$

for all $x > 0$ *and* $T \geq 1$, *where* A, B, C *are positive constants which depend only on the distribution of* X.

This is the KMT result that we were hinting at when discussing (2.6) above. Using this result, in paper [**29**] it is concluded that for the independent uniform–$[0,1]$ distributed random variables $U_1, U_2, \ldots$ one can construct a probability space with a sequence of Brownian bridges $\{\tilde{B}_n(y);\ 0 \leq y \leq 1\}$ on it so that

$$P\left\{\limsup_{n\to\infty} \frac{n^{1/2}}{\log n} \sup_{0\leq y\leq 1} |u_n(y) - \tilde{B}_n(y)| \leq K\right\} = 1, \tag{2.20}$$

where K is a positive absolute constant. In other words, the $O((\log n)n^{-1/2})$ rate of the bridge type strong invariance principle for $u_n(\cdot)$ in (2.20) coincides with that of $\alpha_n(\cdot)$ in (2.17) above. The respective sequences of *Brownian bridges in* (2.17) *and* (2.20) are different by construction, i.e., they *are not the same sequences* of Brownian bridges. On the other hand, just like that of $\alpha_n(\cdot)$ in (2.17), the rate in hand for approximating $u_n(\cdot)$ in (2.20) by a different sequence of Brownian bridges *is* also *best possible* (cf. Remark 4.5.1 of [A1] and Theorem 3.2.2 of [A4]).

The proof of (2.20) is based on the KMT inequality (2.19), and on noting that in terms of the above uniform–$[0,1]$ distributed independent random variables $U_1, U_2, \ldots$ we have

$$(U_{k,n},\ k = 1,\ldots,n) \stackrel{\mathcal{D}}{=} (S_k/S_{n+1},\ k = 1,\ldots,n) \quad \text{for each} \quad n = 1,2,\ldots, \tag{2.21}$$

where $S_n := \sum_{i=1}^{n} \log(1/U_i)$, $n = 1,2,\ldots$, $(U_{k,n},\ k = 1,\ldots,n)$ for each $n = 1,2,\ldots$ are the order statistics of $U_1,\ldots,U_n$, and $\stackrel{\mathcal{D}}{=}$ stands for equality in distribution of the indicated two random vectors.

The sum process

$$R_n(y) := \alpha_n(y) + u_n(y), \quad 0 \leq y \leq 1, \tag{2.22}$$

of the uniform empirical and quantile processes is known in the literature as the (uniform) *Bahadur-Kiefer process* (cf. Bahadur [14], Kiefer [172], [174]). Bahadur [14] introduced R_n as the remainder term in the representation $u_n = -\alpha_n + R_n$. The remainder term R_n, i.e., the Bahadur-Kiefer process, is almost surely smaller asymptotically than the main term α_n, and it enjoys some remarkable asymptotic properties (cf. Bahadur [14], Kiefer [172], [174], and [**185**] for a summary and related new results). One of them in [174] by Kiefer reads as follows:

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4} \sup_{0\leq y\leq 1} |R_n(y)| = 2^{-1/4} \quad \text{a.s.} \tag{2.23}$$

By combining (2.18) and (2.23), Csörgő and Révész in [**29**] observed that the Kiefer process $K_0(y,t) = -K(y,t)$, with $K(y,t)$ as in (2.18), approximates the uniform quantile process u_n as follows:

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4} \sup_{0\leq y\leq 1} |u_n(y) - n^{-1/2}K_0(y,n)| = 2^{-1/4} \quad \text{a.s.} \tag{2.24}$$

In other words, *the same Kiefer* process that KMT [181] constructed for *approximating* α_n as in (2.18) via (2.16), *approximates* u_n *as well* via (2.23).

In view of (2.17) and (2.20) concluding best possible rates of approximation for α_n and u_n respectively via two different sequences of Brownian bridges, it appeared reasonable to write in

[**29**], as well as in [**40**] and Remark 4.5.1 of [A1], that the rate of approximation in (2.24) should probably be far from being best possible. For, indeed, in the light of (2.17) and (2.20) one was inclined to believe that it should be possible to approximate $u_n(\cdot)$ by another Kiefer process better than $K_0(\cdot, n)$ does it in (2.24). Nevertheless Deheuvels [91] showed that *for any Kiefer process* $K_1(\cdot,\cdot)$

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4} \sup_{0\le y\le 1} |u_n(y) - n^{-1/2}K_1(y,n)| > 0 \quad \text{a.s.} \tag{2.25}$$

and, consequently, having $\sup_{0\le y\le 1}|u_n(y) - n^{-1/2}K_1(y,n)| = O(n^{-1/4-\epsilon})$, as $n\to\infty$, almost surely for some $\epsilon > 0$ with any Kiefer process $K_1(\cdot,\cdot)$ is impossible. Thus the rate of convergence in observation (2.24) is optimal not only for the Kiefer process $K_0(\cdot,\cdot) = -K(\cdot,\cdot)$, with $K(\cdot,\cdot)$ as in (2.18), but also for any other Kiefer process $K_1(\cdot,\cdot)$. For a discussion of this matter and further references in this regard we refer to Deheuvels [92].

Standardized empiricals. In their paper [**29**] Csörgő and Révész also posed the problem of studying the LIL behaviour of the *standardized empirical process* $\alpha_n(y)/\sqrt{y(1-y)}$, $0 < y < 1$, and, based on (2.16) and an LIL by Tusnády [290] for a Wiener sheet $W(\cdot,\cdot)$, they gave an answer, but not a complete solution of the problem in hand, by showing that with $\epsilon_n = n^{-1}(\log n)^4$

$$\limsup_{n\to\infty} \sup_{\epsilon_n\le y\le 1-\epsilon_n} \frac{\alpha_n(y)}{(2y(1-y)\log\log n)^{1/2}} = 2^{1/2} \quad \text{a.s.} \tag{2.26}$$

This is to be contrasted with the upper limit 1 replacing $2^{1/2}$ if ϵ_n is replaced by a fixed $\epsilon > 0$. In his fundamental paper [61] of 1977 E. Csáki gave a complete solution of this problem for a wide class of sequences $\{\epsilon_n\}$, $\epsilon_n \downarrow 0$, (cf., e.g., (2.51) in this exposé), and studied a number of related ones as well (cf. also J.A. Wellner [298] in 1978, as well as G.R. Shorack [259] and E. Csáki [62] in 1980). For more details, results and related references on the almost sure and in probability LIL behaviour of the standardized empirical process we refer to [**179**], and to Theorem 5.1.6 and Remark 5.1.1 in [A1].

Studying general quantiles via their uniform versions. Quantiles are revisited in paper [**40**] with Révész, a landmark study of strong approximations of quantile processes in general. In view of the result in (2.20), the study of the uniform quantile process u_n (cf. (2.14)) is continued in [**40**] on the same probability space via the representation of uniform order statistics in (2.21). A complete analogue of (2.15) is established for u_n, which reads as follows: *For the independent uniform–*[0,1] *distributed random variables* $U_1, U_2, \ldots$ *one can construct a probability space with a sequence of Brownian bridges* $\{\tilde{B}_n(y);\ 0\le y\le 1\}$ *on it so that*

$$P\left\{\sup_{0\le y\le 1} |u_n(y) - \tilde{B}_n(y)| > n^{-1/2}(x + A\log n)\right\} \le Be^{-Cx} \tag{2.27}$$

for all $x > 0$*, and integer* $n\ge 1$ *where* A, B, C *are positive absolute constants.*

Naturally, (2.27) implies (2.20). As quoted here, (2.27) is a slightly improved version of its statement in [**40**] and [A1] in that here the restriction of $0 < x \le cn^{1/2}$, $c > 0$, is dropped in favour of all $x > 0$ (cf. Theorem 3.2.1 in [A4] with Horváth). For a preliminary version of (2.27) that was initially used to conclude (2.20), we refer to [**29**].

Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with a distribution function $F(\cdot)$, which is defined to be right continuous. The *empirical distribution function* F_n of the first $n\ge 1$ of these random variables is defined to be right continuous, *à la* $E_n(y)$ of (2.13).

$$F_n(x) := n^{-1} \sum_{i=1}^{n} \mathbb{1}_{(-\infty,x]}(X_i), \quad -\infty < x < \infty, \tag{2.28}$$

or, equivalently, in terms of the order statistics $X_{1,n} \leq \ldots \leq X_{n,n}$ of the random sample $X_1, \ldots, X_n$, as follows

$$F_n(x) := \begin{cases} 0, & \text{if } -\infty < x < X_{1,n}, \\ k/n, & \text{if } \ X_{k,n} \leq x < X_{k+1,n}, \quad 1 \leq k \leq n-1, \\ 1, & \text{if } \ X_{n,n} \leq x < \infty. \end{cases} \tag{2.29}$$

The nth empirical process β_n is defined by

$$\beta_n := n^{1/2}(F_n(x) - F(x)), \quad -\infty < x < \infty. \tag{2.30}$$

Let Q be the quantile function of the distribution function F, defined by

$$Q(y) = F^{-1}(y) := \inf\{x : F(x) \geq y\}, \quad 0 < y \leq 1, \quad Q(0) = Q(0+), \tag{2.31}$$

i.e., Q is the left continuous inverse of the right continuously defined F. Consequently, a random variable X with distribution function F has the same distribution as the random variable $Q(U)$, where U is a uniform–[0,1] random variable, i.e.,

$$X \stackrel{\mathcal{D}}{=} Q(U), \tag{2.32}$$

since we have $P\{Q(U) \leq x\} = P\{U \leq F(x)\} = F(x)$, $-\infty < x < \infty$.

The *empirical quantile function* Q_n is defined to be the left continuous inverse of the right continuously defined empirical distribution function F_n,

$$Q_n(y) = F_n^{-1}(y) := \inf\{x : F_n(x) \leq y\}, \quad 0 < y \leq 1, \quad Q(0) = Q(0+), \tag{2.33}$$

i.e., we have

$$Q_n(y) = F_n^{-1}(y) = \begin{cases} X_{1,n}, & \text{if } \ y = 0, \\ X_{k,n}, & \text{if } \ (k-1)/n < y \leq k/n, \quad 1 \leq k \leq n. \end{cases} \tag{2.34}$$

We note that, unless F has finite support, we have

$$P\left\{ \lim_{n\to\infty} \sup_{0<y<1} |Q_n(y) - Q(y)| = \infty \right\} = 1,$$

i.e., unlike in the case of the uniform quantile process u_n and that of the empirical process β_n, for quantiles in general we do not have a guaranteed Glivenko-Cantelli theorem. Also, only if Q is continuous at $y = y_0$ do we have almost surely that $\lim_{n\to\infty} Q_n(y_0) = Q(y_0)$. Otherwise this statement cannot be true. For further results and comments along these lines we refer to Parzen [209]. In the light of these remarks it is clear that the "natural" general quantile process γ_n that is defined *à la* u_n of (2.14) as

$$\gamma_n(y) := n^{1/2}(Q_n(y) - Q(y)), \quad 0 < y < 1, \tag{2.35}$$

can, at best, be well behaved only at its points of continuity. Hence assume that F is a continuous distribution function. Then Q satisfies

$$Q(y) = F^{-1}(y) = \inf\{x : F(x) = y\}, \quad F(Q(y)) = y, \ \ 0 \le y \le 1, \tag{2.36}$$

and we also have the so-called probability integral transformation

$$F(X) \stackrel{\mathcal{D}}{=} U, \tag{2.37}$$

where U is a uniform–[0,1] random variable, on account of $P\{F(X) \ge y\} = P\{X \ge Q(y)\} = 1 - F(Q(y)) = 1 - y$. Consequently, in this case $U_1 := F(X_1)$, $U_2 := F(X_2), \ldots$ are independent uniform–[0,1] random variables, and the order statistics $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ of the random sample $X_1, \ldots, X_n$ induce the order statistics $U_{1,n} := F(X_{1,n}) \le \cdots \le U_{n,n} := F(X_{n,n})$ of the uniform–[0,1] random sample. Then, the thus induced uniform empirical distribution function E_n of this uniform–[0,1] random sample is (cf. (2.29))

$$\begin{aligned} E_n(y) &= \begin{cases} 0, & \text{if} \ \ 0 \le y < F(X_{1,n}) \\ k/n, & \text{if} \ \ F(X_{k,n}) \le y < F(X_{k+1,n}), \quad 1 \le k \le n-1 \\ 1, & \text{if} \ \ F(X_{n,n}) \le y \le 1 \end{cases} \\ &= F_n(Q(y)), \quad 0 \le y \le 1, \end{aligned} \tag{2.38}$$

and the similarly induced uniform empirical quantile function G_n is given by

$$G_n(y) = E_n^{-1}(y) = \inf\{s : F_n(Q(s)) \ge y\}, \quad 0 < y \le 1, \ \ G_n(0) = G_n(0+), \tag{2.39}$$

i.e., we have

$$\begin{aligned} G_n(y) = F_n^{-1}(y) &= \begin{cases} F(X_{1,n}), & \text{if} \ \ y = 0, \\ F(X_{k,n}), & \text{if} \ \ (k-1)/n < y \le k/n, \quad 1 \le k \le n \end{cases} \\ &= F(Q_n(y)), \quad 0 \le y \le 1, \end{aligned} \tag{2.40}$$

Thus, in terms of $U_i = F(X_i)$, $i = 1, 2, \ldots, n$, we have for any continuous distribution function F

$$\beta_n(Q(y)) = \alpha_n(y), \quad 0 \le y \le 1, \tag{2.41}$$

where α_n is the uniform empirical process as defined in (2.13). Hence, all theorems proved for α_n will hold automatically for β_n as well, simply by letting $y = F(x)$ in (2.41).

Unfortunately, there is no such immediate simple route for transforming γ_n of (2.35) into its own corresponding uniform quantile process

$$\begin{aligned} u_n(y) &= n^{1/2}(G_n(y) - y) \\ &= n^{1/2}(F(Q_n(y)) - y), \quad 0 \le y \le 1. \end{aligned} \tag{2.42}$$

A first step is via using the mean value theorem to write

$$\begin{aligned} \gamma_n(y) &= n^{1/2}(Q_n(y) - Q(y)) \\ &= n^{1/2}(Q(F(Q_n(y))) - Q(y)) \\ &= n^{1/2}(F(Q_n(y)) - y)(1/f(Q(\theta_n(y)))) \\ &= n^{1/2}(G_n(y) - y)(1/f(Q(\theta_n(y)))) \\ &= u_n(y)(1/f(Q(\theta_n(y)))), \quad 0 < y < 1, \end{aligned} \tag{2.43}$$

where $G_n(y) \wedge y < \theta_n(y) < G_n(y) \vee y$, $y \in (0,1)$, $n = 1, 2, \ldots$, provided of course that we have $Q'(y) = 1/f(Q(y)) < \infty$ for $y \in (0,1)$, i.e., provided that F is an absolutely continuous distribution function (with respect to Lebesgue measure) with a strictly positive density function $f = F'$ on the real line. The function $f(Q(\cdot))$ is called the *density-quantile function* and $Q'(\cdot) = 1/f(Q(\cdot))$ the *quantile-density function* by Parzen [206], [208]. For estimating the quantile-density function we refer to [**123**] with Deheuvels and Horváth. For a recent view and review of quantiles by Parzen, we refer to [211] in [V2].

The relationship (2.43) shows that if we expect to have the quantile process γ_n behave in the same way as the uniform quantile process u_n does, then we should at least assume that F has a strictly positive density function on the real line. Moreover, (2.43) also shows that, for the sake of comparing γ_n with its associated u_n as in (2.42), one should multiply the former by the density-quantile function $f(Q)$. Hence, on assuming that $f = F'$ exists on the real line, in their paper [**40**], Csörgő and Révész define the *general quantile process* ρ_n by

$$\rho_n(y) := n^{1/2} f(Q(y))(Q_n(y) - Q(y)), \quad 0 \le y \le 1. \tag{2.44}$$

For further details on the train of thought on quantiles leading up to this definition of the general quantile process, in addition to paper [**40**], we refer to related sections of the books [A1], [A2], [A3], [A4], Shorack and Wellner [263], and paper [**73**].

We will now see that studying ρ_n via its own u_n as in (2.42) will also result in conveniently comparing ρ_n with α_n of (2.41) in the Bahadur [14] and Kiefer [172], [174] sense as well. On the other hand, owing to the presence of the density-quantile function in its definition, ρ_n as in (2.44) does not lend itself readily to constructing confidence bands for the quantile function Q. In this regard we refer to paper [**62**] with Révész and Chapter 4 of book [A2]. Having only confidence bands for Q in mind, it may be better to start with $\alpha_n(y)$ of (2.41). Assuming only that F is continuous, via this $\alpha_n(y) = \beta(Q(y))$, $0 \le y \le 1$, one can, for example, easily arrive at (cf. [**112**], or [**113**]; these two listings are identical in content)

$$\begin{aligned} &\lim_{n\to\infty} P\left\{ Q_n(y_n^{-1/2} c(\alpha)) \le Q(y) \le Q_n(y + n^{-1/2} c(\alpha)), \epsilon_n \le y \le 1 - \epsilon_n \right\} \\ &\qquad = P\left\{ \sup_{0 \le y \le 1} |B(y) \le c(\alpha) \right\} = 1 - \alpha, \end{aligned} \tag{2.45}$$

where $B(\cdot)$ is a Brownian bridge, $c(\alpha)$ is a positive real number for which we have the latter equality hold true for $\alpha \in (0,1)$, and $\{\epsilon_n, n \ge 1\}$ is any sequence of real numbers such that $\epsilon_n \to 0$ and $n^{1/2}\epsilon_n \to \infty$, as $n \to \infty$.

Now, on account of (2.43) and (2.44), with $\theta_n(y)$ as in (2.43), we have

$$\rho_n(y) = u_n(y)(f(Q(y))/f(Q(\theta_n(y)))), \quad 0 < y < 1. \tag{2.46}$$

Hence it appears to be reasonable to hope for an asymptotic theory of ρ_n that would resemble that of u_n if one could only "regulate" the ratio $f(Q(y))/f(Q(\theta_n(y)))$ uniformly in $y \in (0,1)$. As a first step in this direction, Csörgő and Révész in [**40**] show that if

(i) *F is twice differentiable on (a, b), where*

$$a = \sup\{x : F(x) = 0\}, \quad b = \inf\{x : F(x) = 1\}, \quad -\infty \le a < b \le \infty,$$

(ii) $F'(x) = f(x) > 0$, $x \in (a, b)$, *and*

(iii) *for some* $\gamma > 0$ *we have*

$$\sup_{0<y<1} y(1-y)|f'(Q(y))|/f^2(Q(y)) \le \gamma, \tag{2.47}$$

then

$$\frac{f(Q(y_1))}{f(Q(y_2))} \le \left\{ \frac{y_1 \vee y_2}{y_1 \wedge y_2} \, \frac{1 - (y_1 \wedge y_w)}{1 - (y_1 \vee y_2)} \right\}^\gamma \tag{2.48}$$

for every pair $y_1, y_2 \in (0, 1)$.

In the literature on nonparametric statistics, it is customary to define the so-called *score function* (cf., e.g., Hájek and Šidák [142], p. 19):

$$J(y) := -f'(Q(y))/f(Q(y)) = -\frac{d}{dy} f(Q(y)). \tag{2.49}$$

Thus, condition (iii) (2.47) can be written as

$$\sup_{0<y<1} y(1-y)|J(y)|/f(Q(y)) \le \gamma. \tag{2.50}$$

For examples and a discussion of tail monotonicity assumptions of extreme value theory as related to (2.50), we refer to Parzen [208], [209].

The score function J of (2.49) plays an important role in nonparametric statistics in general, and robust statistical analysis in particular (cf., e.g., Hájek and Šidák [142], and Huber [151]). Owing to its importance, and because of our lack of knowledge of f in most practical situations, it is desirable to estimate J, given a random sample on F. For results on estimating J, and for further discussions along these lines, we refer to Hájek and Šidák ([142], p. 259), Parzen [208], Chapter 10 of [A2], [**76**] with Révész, and Burke and Horváth [41].

Returning now to the problem of the general quantile process ρ_n versus its uniform version u_n (cf. (2.42), when comparing these two processes in [**40**], Csörgő and Révész use a Csáki-type law of the iterated logarithm for a uniform quantile process in combination with condition (iii) (2.47) and its consequence (2.48). As mentioned already in view of the problem posed in [**29**] in connection with (2.26), Csáki [61] studied the latter for a wide class of sequences $\{\epsilon_n\}$, $\epsilon_n \downarrow 0$. We quote here one special case, which reads as follows: *With* $\epsilon_n = dn^{-1} \log\log n$ *and* $d \ge 0.236\ldots$, *we have*

$$\limsup_{n\to\infty} \sup_{\epsilon_n \le y \le 1-\epsilon_n} \frac{|\alpha_n(y)|}{(2y(1-y)(\log\log n)^{1/2}} = 2^{1/2} \quad \text{a.s.} \tag{2.51}$$

Based on this result, in [**40**] Csörgő and Révész conclude

$$\limsup_{n\to\infty} \sup_{\delta_n \le y \le 1-\delta_n} \frac{|u_n(y)|}{(2y(1-y)\log\log n)^{1/2}} \le 2 \cdot 2^{1/2} \quad \text{a.s.} \tag{2.52}$$

with $\delta_n = 25n^{-1} \log\log n$.

We note in passing that in their Theorem 16.4.1 in [263], Shorack and Wellner establish (2.52) with upper bound 2 instead of $2 \cdot 2^{1/2}$ and 9 replacing 25 in the definition of δ_n.

Using (2.52) in combination with (2.48), paper [**40**] viewed via [**64**] concludes (cf. also Theorem 3.2.1 in [A2]) the following basic results: *Let* ρ_n *and* u_n *be respectively defined in terms of*

$X_{k,n}$ and $U_{k,n} = F(X_{k,n})$ as in (2.44) *and* (2.42), *and assume the conditions* (i), (ii) *and* (iii) (2.47). *Then, as $n \to \infty$,*

$$\sup_{1/(n+1) \le y \le n/(n+1)} |\rho_n(y) - u_n(y)| \stackrel{a.s.}{=} \begin{cases} O(n^{-1/2}(\log\log n)^{1+\gamma}), & \text{if} \quad \gamma \le 1 \\ O(n^{-1/2}(\log\log n)^{\gamma}(\log n)^{(1+\epsilon)(\gamma-1)}), & \text{if} \quad \gamma > 1, \end{cases} \tag{2.53}$$

for all $\epsilon > 0$. Moreover, if in addition to (i), (ii) *and* (iii) (2.47), *we also assume*

(iv) $\lim_{y \downarrow 0} f(Q(y)) > 0$ *and* $\lim_{y \uparrow 1} f(Q(y)) > 0$, *both finite,* (2.54)

or

(v) *if* $\lim_{y \downarrow 0} f(Q(y)) = 0$, *then f is non-decreasing in a right-neighbourhood of $Q(0) = Q(0+)$, and if* $\lim_{y \uparrow 1} f(Q(y)) = 0$, *then f is non-increasing in a left-neighbourhood of $Q(1)$,*

then, as $n \to \infty$,

$$\sup_{0 \le y \le 1} |\rho_n(y) - u_n(y)| \stackrel{a.s.}{=} O(n^{-1/2} \log\log n) \tag{2.55}$$

if (iv) *obtains, and*

$$\sup_{0 \le y \le 1} |\rho_n(y) - u_n(y)| \stackrel{a.s.}{=} \begin{cases} O\left(n^{-1/2} \log\log n\right), & \text{if} \quad \gamma < 1 \\ O\left(n^{-1/2}(\log\log n)^2\right), & \text{if} \quad \gamma = 1 \\ O\left(n^{-1/2}(\log\log n)^{\gamma}(\log n)^{(1+\epsilon)(\gamma-1)}\right) & \text{if} \quad \gamma > 1 \end{cases} \tag{2.56}$$

for all $\epsilon > 0$, if (v) *obtains.*

It is somewhat surprising that, in general, one needs an extra condition for going from approximating ρ_n by u_n over the intervals $[1/(n+1), n/(n+1)]$ (cf. (2.53)) to approximating ρ_n by u_n on the unit interval $[0, 1]$ (cf. (2.55) and (2.56)). Parzen [208] gave an example, via letting $1 - F(x) = \exp(-x - C \sin x)$, $x \ge 0$, $1/2 < C < 1$, in which case the conditions (i), (ii), (iii) (2.47) are satisfied, the latter with $\gamma = 1/(1 - c)$. Hence, we have (2.53) with $\gamma = 1/(1 - C)$. On the other hand, the conditions (iv), (v) (2.54) fail to hold. Nevertheless (cf. [**64**]), in case of this example, we still have (2.56) as well, with $\gamma = 1/(1 - C)$. Thus, for results like (2.55) and (2.56), conditions (iv), (v) (2.54) are not necessary. For further details we refer to [**64**], and to Section 5.4 of [A1].

In addition to [**40**], [**64**] for various versions of proofs, and further results and comments on estimating the distance of ρ_n from u_n, we refer to Chapters 3–7 of [A1], [A2], Chapter 6 of [A4], and to Sections 16.3, 16.4, 18.1 and 18.2 of Shorack and Wellner [263]. The proof of Theorem 18.2.1 in the latter book results in a slightly better rate than that of (2.56) in case of $\gamma > 1$. Namely, if $\gamma > 1$, then the exponent of the $\log\log n$ term of the rate function in Theorem 18.2.1 of [263] is 1 instead of γ as in (2.56).

However, as we will now see, the results in (2.53), (2.55) and (2.56) do not leave much room for improvement, for on page 153 of [A1] it is shown that, if in addition to the conditions for

(2.55) and (2.56), one assumes also that f is twice differentiable with $|f"|$ bounded above, and $|f'|$ bounded away from zero on a finite interval $(\bar{a}, \bar{b}) \subset (a, b)$, with (a, b) as in (i) (2.47), then

$$\limsup_{n\to\infty} \frac{n^{1/2}}{\log\log n} \sup_{\bar{a}\le y\le \bar{b}} |\rho_n(y) - u_n(y)| > K_F > 0 \quad \text{a.s.} \tag{2.57}$$

for some constant K_F depending on F.

This result is to be compared to having also

$$\limsup_{n\to\infty} \frac{n^{1/2}}{\log\log n} \sup_{\delta_n\le y\le \delta_n} |\rho_n(y) - u_n(y)| < K \quad \text{a.s.} \tag{2.58}$$

under the conditions (i), (ii) and (iii) (2.47), where $\delta_n = 25n^{-1}\log\log n$ (cf. (2.52)) and the constant K depends only on the value of γ in (2.48) (cf. (3.3) in [**40**], or (4.5.11) in [A1]). A best available version of (2.58) is that of (18.2.9) of Shorack and Wellner [263] with $K = \gamma 2^{\gamma+3}$ and $\delta_n = 9n^{-1}\log\log n$. Whichever way, (2.58) is seen to be best possible in the "middle" in the sense of (2.57).

We note also that if we want to estimate the distance between u_n as in (2.42) and ρ_n as in (2.44) only in probability, then, under the conditions (i), (ii) and (iii) (2.47), as $n \to \infty$, we have (cf. Theorem 6.1.5 of [A4] with Horváth):

$$\sup_{1/(n+1)\le y\le n/(n+1)} |\rho_n(y) - u_n(y)| = O_P(n^{-1/2}\log\log n). \tag{2.59}$$

Commenting on (iii) (2.47), Parzen [209] (cf. [**64**], Sections 5.3, 5.4 of [A2], and page 651 of Shorack and Wellner [263]) shows that *if* $g := f(Q)$ *and*

$$\lim_{y\to 0 \text{ or } 1} y(1-y)\frac{f'(Q(y))}{f^2(Q(y))} = \lim_{y\to 0 \text{ or } 1} y(1-y)\frac{g'(y)}{g(y)} = \begin{cases} a_0 & \text{at } 0 \\ a_1 & \text{at } 1, \end{cases} \tag{2.60}$$

then

$$y^{-a_0}g(y) \text{ and } y^{-a_1}g(1-y) \text{ are both slowly varying at 0.} \tag{2.61}$$

Moreover, if $a_0 \vee a_1 \le 0$*, then all absolute moments are finite, while if* $a_0 \vee a_1 > 0$*, then*

$$E|X|^r < \infty \text{ if } r < 1/(a_0 \vee a_1). \tag{2.62}$$

Studying (iii) (2.47), Mason [195] shows that *if* $Q = F^{-1}$ *is continuous, then for each* r_1, $r_1 > 0$ *one has*

$$\lim_{n\to\infty} \sup_{0\le y\le 1} \left| y^{r_1}(1-y)^{r_2}(Q_n(y) - Q(y)) \right| = \begin{cases} 0 & \text{a.s. } \textit{if } E(X^-)^{1/r_1} < \infty \textit{ and } E(X^+)^{1/r_2} < \infty \\ \infty & \text{a.s. } \textit{if either of these expectations equals } \infty, \end{cases} \tag{2.63}$$

a Glivenko-Cantelli theorem for the general quantile process when properly "curbed" on the tails.

In the light of having (2.55) and (2.56) under the conditions (i), (ii), (iii) (2.47) and (iv), (v) (2.54) combined, as $n \to \infty$, we of course have as well

$$\sup_{0\le y\le 1} |n^{-1/2}\rho_n(y)| = \sup_{0\le y\le 1} |f(Q(y))(Q_n(y) - Q(y)| \to 0 \quad \text{a.s.,} \tag{2.64}$$

a Glivenko-Cantelli theorem for $n^{-1/2}\rho_n$.

In view of (2.60)–(2.64), Mason [195] also discusses the relationship between (2.63) and (2.64), and concludes (2.64) *à la* (2.63) as follows: *Assume that F has density $f > 0$ with a continuous derivative on the support (a, b) of F* (cf. (i), (ii) (2.47)). If for some $0 < \gamma_1 < \infty$ and $0 < \gamma_2 < \infty$

$$\lim_{y\downarrow 0} y\frac{g'(y)}{g(y)} = \gamma_1 \quad \text{and} \quad \lim_{y\uparrow 1}(-1)(1-y)\frac{g'(y)}{g(y)} = \gamma_2 \tag{2.65}$$

with $g = f(Q)$ as in (2.60), then

$$\lim_{n\to\infty} \sup_{0\le y\le 1} |n^{-1/2}\rho_n(y)| \to 0 \quad \text{a.s.} \tag{2.66}$$

For further results and comments along these lines we refer to [**40**], [**64**], [A1] and [A4].

The results in (2.53), (2.55) and (2.56) on approximating ρ_n by u_n can be transformed into Gaussian approximations of ρ_n via (2.27) and (2.24) respectively, which reads as follows (cf. also Theorem 3.2.4 in [A2]): *Let ρ_n and u_n be respectively defined in terms of $X_{k,n}$ and $U_{k,n} = F(X_{k,n})$ as in* (2.44) *and* (2.42)*, and assume the conditions* (i), (ii) *and* (iii) (2.47) *that led to concluding* (2.53). *Then, on the probability space for* (2.27) *with the same sequence of Brownian bridges $\{\tilde{B}_n(y);\, 0 \le y \le 1\}$ as in there, in view of* (2.53) *we have, as $n \to \infty$,*

$$\begin{aligned}&\sup_{1/(n+1)\le y\le n/(n+1)} |\rho_n(y) - \tilde{B}_n(y)| \\ &\quad \overset{a.s.}{=} \begin{cases} O(n^{-1/2}\log n), & \text{if } \gamma < 2 \\ O(n^{-1/2}(\log\log n)^{\gamma}(\log n)^{(1+\epsilon)(\gamma-1)}), & \text{if } \gamma \ge 2, \end{cases}\end{aligned} \tag{2.67}$$

where $\gamma > 0$ is as in (2.48) *and $\epsilon > 0$ is arbitrary. Also, under the same conditions, with the Kiefer process $\{K_0(y,t);\, 0 \le y \le 1,\, t \ge 0\}$ as in* (2.24)*, we have*

$$\begin{aligned}&\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4} \sup_{1/(n+1)\le y\le n/(n+1)} |\rho_n(y) - n^{-1/2}K_0(y,n)| \\ &\quad \overset{a.s.}{=} 2^{-1/4}.\end{aligned} \tag{2.68}$$

If, in addition to conditions (i), (ii), (iii) (2.47)*, we also assume* (iv), (v) (2.54)*, then, as $n \to \infty$, in view of* (2.55)*,*

$$\sup_{0\le y\le 1} |\rho_n(y) - \tilde{B}_n(y)| \overset{a.s.}{=} O(n^{-1/2}\log n) \tag{2.69}$$

if (iv) *obtains, and if* (v) *obtains, then in view of* (2.56)*,*

$$\sup_{0\le y\le 1} |\rho_n(y) - \tilde{B}_n(y)| \overset{a.s.}{=} \begin{cases} O(n^{-1/2}\log n), & \text{if } \gamma < 2 \\ O(n^{-1/2}(\log\log n)^{\gamma}(\log n)^{(1+\epsilon)(\gamma-1)}), & \text{if } \gamma \ge 2, \end{cases} \tag{2.70}$$

where $\gamma > 0$ is as in (2.48) *and $\epsilon > 0$ is arbitrary. Moreover, on account of* (2.24), (2.55) *and* (2.56)*, if any one of* (iv), (v) *of* (2.54) *obtains, then*

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4} \sup_{0\le y\le 1} |\rho_n(y) - n^{-1/2}K_0(y,n)| \overset{a.s.}{=} 2^{-1/4}. \tag{2.71}$$

In view of result (2.25) by Deheuvels [91], the almost sure rate of convergence in (2.71) is, just like that of (2.24), best possible. On account of the rate of approximation of u_n by a sequence of Brownian bridges $\tilde{B}_n$ as in (2.20) being best possible, so is also the rate of approximation of ρ_n by the same sequence of Brownian bridges $\tilde{B}_n$ in (2.69) and that of (2.70) for $\gamma < 2$. The rate of approximation in (2.70) when $\gamma \geq 2$ is also best possible in terms of $n^{-1/2}$, and there does not seem to be much room for improvement for the powers of the $\log n$ and $\log\log n$ terms. It would however be desirable to have a version of the coupling inequality (2.27) for ρ_n and $\tilde{B}_n$ as well. **Extensions of Kiefer's asymptotic theory to the general Bahadur-Kiefer process**.The *quantile papers* [**29**], [**40**] and [**64**], together with G.R. Shorack [260], also conclude a *randomized Gaussian (iterated Kiefer process) interpretation for the Bahadur-Kiefer process* (cf. R.R. Bahadur [14]), J. Kiefer [172], [174] and Theorem 4.1 of [**167**] with B. Szyszkowicz) of uniformly distributed random variables via the KMT theorems in [181] and [182], and those of A.H.C. Chan [44]. For developments on the problem of best possible approximation of the uniform quantile process by a Kiefer and iterated Kiefer processes we refer to P. Deheuvels [91], [92], [93]. We note that papers [**40**] and [**64**] also amount to a major *extension of Kiefer's asymptotic theory of the uniform Bahadur-Kiefer process to the general Bahadur-Kiefer process.* For example, a combination of Kiefer's result in (2.23) respectively with (2.53), (2.55) and (2.56) leads to comparing in sup-norm the general quantile process ρ_n as in (2.44) under the respective conditions of (2.53), (2.55) and (2.56) to the uniform empirical process α_n as in (2.41). For further details and results along these lines we refer to [A1], the 1983 SIAM monograph [A2], book [A4], the papers [**64**], [**167**], [**188**], and Section 4.2 in Csáki et al. [69] in [V2]. For an L^p–view of Bahadur-Kiefer processes we refer to [**183**] and [**195**] with Zhan Shi (cf. Theorem 4.4 in [69]). **Path increments of Brownian motion and partial sums: a scenic route from Erdős–Rényi laws to LIL**.*Back to the initial stages of strong approximations of partial sums by a Wiener process* and their *interplay with studying fine analytic path properties of increments of Brownian motion*, we quote again from [**186**] by Miklós Csörgő in [B3].

"In connection with (2.5) and (2.6) above, we mentioned how a result of P. Bártfai [15] had played a crucial role in settling the question of what the lower limit to the strong invariance principle for partial sums of random variables should be like. On the other hand, the Erdős-Rényi [117] new law of large numbers produced a new and direct proof of Bártfai's just mentioned theorem (cf. *Proof of Theorem 2.3.1* in book [A1]) and thus had also given a tremendous insight into the nature of what *strong invariance* versus *strong measure determining noninvariance principles* are all about. Moreover, pivoting this fresh insight against the first strong invariance principle for partial sums (cf. V. Strassen [272]), one realized that there must be a scale of theorems in between Strassen's LIL, the Erdős-Rényi 'large increment' laws, and beyond. Moreover, the celebrated Paul Lévy *moduli of continuity* results for Brownian motion (Wiener process) suggested a duality for these *large and small increment laws.* This is the kind of thinking that is embodied for example in Lemmas 1.1.1 and 1.2.1 in book [A1], and which has earlier guided us to the first two path-breaking papers [**43**] and [**44**] in this regard.

A preliminary version of the main result of paper [**43**] of 1979 is already quoted in their presentation in Oberwolfach, March 28-April 3 1976, by M. Csörgő and A.H.C. Chan [**34**] (cf. Theorem G of the latter). This also indicates that we began our work on these topics just about at the time of publishing our first two papers [**27**] and [**28**] in 1975 together. Moreover, it was our work on the paper [**43**] of 1979 that has inspired and led to the already mentioned Ph.D. thesis of A.H.C. Chan [44] whose theorems on the Wiener sheet and the Kiefer process are then quoted in Chapter 1 of book [A1] of 1981. This sequence of events and results have, in turn, also led to many further important results in G.R. Shorack [260], W. Stute [274], and D.M. Mason, G.R. Shorack and J.A. Wellner [199] on strong oscillation theorems for the uniform empirical and quantile processes. For further contributions along these lines we refer to [**167**], and to P. Deheuvels [90], [91], [92], [93], where he also concludes that the J. Kiefer [174] and KMT [181] based strong approximation of the uniform quantile process in paper [**29**] is best possible (cf. (2.25)).

Back to the papers [**43**] and [**44**], *per se* they have *initiated the contributions* of the Hungarian school *to studying fine analytic path properties of increments of Brownian motion and related stochastic processes*. Thus they deserve to be singled out as fundamental first steps in this regard. They have also played an important seminal role in many papers ever since their appearance. The respective titles of the further well known papers [**38**], [**39**], [71], and [237] at the end of the seventies speak for themselves, and so does also that of [49] in [B3]. The paper [71] in 1979 also signals the beginning of Pál Révész's illustrious collaboration with Endre Csáki. Starting with C.M. Deo [99] and S.A. Book and T.R. Shore [32], the literature on and around the topics of papers [**43**] and [**44**] has grown vast. For a glimpse of it we refer to Endre Csáki (1988, Doctoral Dissertation), Csáki et al. [69] in [V2], B. Chen [49] in [B3], and [48]. The way it all began in paper [**43**] looks like this:

Let a_T be a monotonically non-decreasing function of T such that

(i) $0 < a_T \le T$

(ii) T/a_T *is monotonically non-decreasing.*

Define

$$\beta_T := \left(2a_T\left(\log\frac{T}{a_T} + \log\log T\right)\right)^{-1/2}.$$

Then

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\beta_T|W(t+a_T)-W(t)| = 1, \quad a.s.,$$

$$\begin{aligned}&\limsup_{T\to\infty}\beta_T|W(t+a_T)-W(t)|\\ &\quad = \limsup_{T\to\infty}\sup_{0\le s\le T-a_T}\beta_T|W(T+s)-W(t)| = 1, \quad a.s.,\end{aligned}$$

and

$$\limsup_{T\to\infty}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}\beta_T|W(T+s)-W(t)| = 1, \quad a.s.,$$

If we also have

(iii) $$\lim_{T\to\infty}\log(T/a_T)/\log\log T = \infty,$$

then $\limsup_{T\to\infty}$ *can be replaced by* $\lim_{T\to\infty}$ *in the above statements.*

Choosing, for example, a_T as $c\log T$, cT and 1 respectively, we get:

(a) For any $c > 0$, the Erdős-Rényi [117] law for Brownian motion

$$\lim_{T\to\infty}\sup_{0\le t\le T-c\log T}\frac{|W(t+c\log T)-W(t)|}{c\log T} = \sqrt{\frac{2}{c}} \quad \text{a.s.}$$

(b) For $0 < c \leq 1$, the LIL

$$\limsup_{T\to\infty} \sup_{0\leq t\leq T-cT} \frac{|W(t+cT)-W(t)|}{(2cT\log\log T)^{1/2}} = 1 \quad \text{a.s.},$$
$$\limsup_{T\to\infty} \sup_{0\leq t\leq T-cT} \sup_{0\leq s\leq cT} \frac{|W(t+s)-W(t)|}{(2cT\log\log T)^{1/2}} = 1 \quad \text{a.s.},$$

and

(c) $$\lim_{T\to\infty} \sup_{0\leq t\leq T-1} \frac{|W(t+1)-W(t)|}{(2\log T)^{1/2}} = 1 \quad \text{a.s.}$$

With $c = 1$ in (b) here, we get the classical LIL for Brownian motion, while (a) is a special case of the general Erdős-Rényi [117] law. As in Chapter 1 of book [A1], in [**43**] the authors also deduce P. Lévy's uniform modulus of continuity for the Wiener process.

The original general version of the Erdős–Rényi [117] law reads as follows: *Let $X_1, X_2, \dots$ be independent identically distributed random variables with mean zero and a moment generating function $R(t) := Ee^{tX_1}$ that is finite in a neighbourhood of $t = 0$. Define*

$$\rho(x) := \inf_t e^{-tx} R(t), \tag{2.72}$$

the so-called Chernoff function of X_1. Then, for any $c > 0$, as $n \to \infty$, we have

$$\max_{0\leq k\leq n-[c\log n]} \frac{S(k+[c\log n])-S(k)}{[c\log n]} \longrightarrow \alpha(c), \tag{2.73}$$

where

$$\alpha(c) := \sup\{x : \rho(x) \geq e^{-1/c}\}. \tag{2.74}$$

Moreover, the function $\alpha(c)$, $c > 0$, uniquely determines the moment generating function, and hence also the distribution function, of X_1.

For developments in the seventies on Erdős–Rényi laws we refer to J. Komlós and G. Tusnády [183], S.A. Book [31], and S. Csörgő [75].

If we open the window wider than $a_n = [c\log n]$ in (2.73), for example so wide that $a_n/\log n \to \infty$ as $n \to \infty$, then we see more than one single distribution, strong invariance takes over from the Erdős–Rényi law and all partial sums with a moment generating function will behave like Brownian motion as a consequence of combining the just quoted results of paper [**43**] with KMT [181], [182]. For further results along these lines when the existence of the moment generating function is not assumed, we refer to exposition [**43**], Chapter 3 of book [A1], and Q-M. Shao [256].

In paper [**44**] of 1979 that in our thoughts evolved just about concurrently with [**43**] of 1979, and [**38**] and [**39**] of 1978, the following *modulus of non-differentiability is proved for the Wiener process*

$$\lim_{h\downarrow 0} \inf_{0\leq t\leq 1-h} \sup_{0\leq s\leq h} \sqrt{8\log\frac{h^{-1}}{\pi^2 h}}\,|W(t+s)-W(t)| = 1 \quad \text{a.s.} \tag{2.75}$$

and, studying the same problem over long time intervals, we arrive at

$$\liminf_{T\to\infty} \gamma_T \inf_{0\leq t\leq T-a_T} \sup_{0\leq s\leq a_T} |W(t+s)-W(t)| = 1 \quad \text{a.s.}, \tag{2.76}$$

where $0 < a_T \leq T$, a_T/T *is non-increasing, and*

$$\gamma_T := \left(\frac{8(\log(T/a_T) + \log\log T)}{\pi^2 a_T} \right)^{1/2}. \tag{2.77}$$

If, as $T \to \infty$, *we also have* $\log(T/a_T)/\log\log T \uparrow \infty$, *then* $\liminf_{T\to\infty}$ *can be replaced by* $\lim_{T\to\infty}$ *in* (2.76).

For examples with a_T taking various values, we refer to Chapter 1 of book [A1]. In particular, if $a_T = T$, then (2.76) reduces to *the* so-called *other LIL* of K.L. Chung [56] when it is applied to the Wiener process. The relationship of the result of (2.75) to a related result of A. Dvoretzky [106] and that of S.J. Taylor [285] is discussed on page 85 of book [A1].

On combining these results of exposition [**43**] for the Wiener process with KMT [181], [182] and A.A. Mogul'skiĭ [201], one obtains corresponding results for similar $\min_{1\leq n\leq N-a_N} \max_{1\leq k\leq a_N} |S(n+k) - S(n)|$ increments of partial sums of i.i.d. random variables. In particular, we thus *obtain the other LIL* of K.L. Chung (cf. also Section 3.3 of book [A1]) for partial sums of i.i.d. random variables *that* are assumed to *have only two moments*, a result that was first proved to be best possible under the latter condition by N.C. Jain and W.E. Pruitt [158]. In connection with the **Conjecture** on page 122 of book [A1] that parallels the Erdős–Rényi [117] law via $\min_{1\leq n\leq a_N} \max_{1\leq k\leq a_N} |S(n+k) - S(n)|$, we refer to the insightful exposition by E. Csáki and A. Földes [65]."

For closely related developments we refer to [**39**] with A.H.C. Chan and P. Révész, Révész [237], Bin Chen [48], [**181**] with B. Chen, and W. Wang [296].

For further reviews and recent advances on path properties of stochastic processes we refer to the first five papers of [V2]. Csáki et al. [69] initiate their discussion with quoting from [**43**]. The second one in **Part 1**, [171] by Davar Khoshnevisan, also relates to [**44**], [**50**] and [A1].

The book [A1] with Pál Révész was already mentioned and referred to many times in our discussion. Writing about the eighties in his tribute to Pál Révész, in [**186**] Miklós Csörgő reminisces:

"Let me ... mention ... that we were glad to see book [A1] appear in 1981. On and off, working on it, we were lucky to be able to spend a substantial part of the seventies together, a most enjoyable period of our life, full with labouring aspiration and drive to succeed in our task."

In view of our survey of this period so far, and recalling also Kiefer's already mentioned review of [**27**], [**28**] and KMT [181], we note that Kiefer's foresighted provision for the future of this approach has come true. After the appearance of these first landmark papers, contributions by mathematicians–probabilists–statisticians world-wide have made the *Hungarian construction* school international, and it continues to play a prominent role in the theory and applications of strong and weak invariance principles. Here we mention only the books and review papers in this regard [A1], [A2], [A3], [A4], [A5], [263], [242], [**63**], [**73**], [**193**] and, for further information, refer to the references therein. For example, there are more than 200 references up to 1993 that are made use of in [A4]. We continue with quoting from the **Preface** of [A4]:

> The 1981 book of Csörgő and Révész, "Strong Approximations in Probability and Statistics", reflects the birth and development of the *Hungarian construction* for proving invariance principles for partial sums, empirical and quantile, as well as for some related processes. It also studies the fine analytic properties of the approximating Gaussian processes on their own, as well as for the sake of deriving new

invariance principles for the approximated processes. This approach continues to play an important role in the theory and applications of strong and weak invariance principles. Vigorous and far reaching developments have taken place in these areas since the appearance of this book and the first fundamental papers it is based on. We are pleased to say that the *Hungarian construction* school has since become international, with many outstandingly significant contributions by mathematicians from all over the world to the consolidation and manifold extensions of these tools for strong and weak approximations.

For further glimpses of the interplay of Gaussian processes and strong approximations we refer to [**46**], [**51**] to the review of the Shorack and Wellner book [263] by Miklós Csörgő [**193**], and to the books [V1] and [B3] (cf. also Csáki et al. [69] in [V2]). Here we continue our presentation of this interplay in a historical context in the first half of the eighties with occasional glimpses beyond.

Révész's paper [239] of 1982 establishes the *first and best possible rates of convergence for* the above quoted *CsR large increments* of a Wiener process of paper [**43**]. This, *in turn*, inspired [**52**] with Josef Steinebach (we were lucky to be privy to a preliminary version of [239] of 1982 at Carleton University), where *the first rate of convergence is proved for the Erdős–Rényi* [**117**] *laws* of large numbers. Consequently, *exact rates of convergence* were established for the latter laws, together with that of L.A. Shepp [258], in various contexts of generalities by P. Deheuvels, L. Devroye and J. Lynch [95], P. Deheuvels and L. Devroye [94]. For further related contributions along these lines we refer to A. de Acosta and J. Kuelbs [89], J. Steinebach [269], and D.M. Mason [196]. For some details on these results we refer to [**189**]. For more recent developments along these lines we refer to Andrei N. Frolov, *Teor. Veroyatnost. i Primenen.* **48** (2003), no. 1, 104–121.

Best possible approximations of random walk local time by Brownian local time. Path properties. Our next topic is *random walk local time via invariance principles.* Let $S(0) = 0$, $S(k) = X_1 + \cdots + X_k$, $k \geq 1$, be a simple symmetric *random walk*, define its *local time* (site)

$$\xi(x, n) := \#\{k : 1 \leq k \leq n,\ S(k) = x\}, \quad x = 0, \pm 1, \pm 2, \ldots, \tag{2.78}$$

and recall *Paul Lévy's notion of Brownian local time*, $\{L(x,t);\ x \in \mathrm{R}^1, t \geq 0\}$, via the *occupation time*

$$H(A, t) := \lambda\{s : 0 \leq s \leq t,\ W(s) \in A\} = \int_A L(x,t)dx, \tag{2.79}$$

for any $t > 0$ and Borel set A of the real line R^1, where $\lambda(\cdot)$ is the Lebesgue measure and $\{W(t);\ t \geq 0\}$ is standard Wiener process. H.F. Trotter [288] proved that the occupation time *random measure* $H(A,t)$ of $W(\cdot)$ *is almost surely absolutely continuous with respect to* $\lambda(\cdot)$ and that its *Radon-Nikodym derivative* $L(x,t)$ in (2.79), the local time of $W(\cdot)$, is *continuous in both arguments.* The papers by K.L. Chung and G.A. Hunt [57], Ch. J. Stone [270], H. Kesten [167], as well as the monographs by K. Itô and H.P. McKean Jr. [156], and F.B. Knight [179] show impressively many ways that the asymptotic behaviour of $\xi(\cdot,\cdot)$ and $L(\cdot,\cdot)$ is similar. In [238] Révész establishes *the first strong invariance principle* in this regard, *with a rate of convergence via* an appropriate *Skorohod-type construction*, which concludes that *on a rich enough probability space, as* $n \to \infty$,

$$\sup_x |\xi(x,n) - L(x,n)| = o(n^{1/4+\epsilon}) \quad a.s. \tag{2.80}$$

for any $\epsilon > 0$, where sup *is taken over all integers.*

Extending the latter result via Skorohod construction [72] with E. Csáki to the case of an integer-valued recurrent random walk, where various moments of X_1 are assumed to exist, the authors also note that $o(n^{1/4+\epsilon})$ of *Révész 's result in* (2.80) *cannot be replaced by* $o(n^{1/4})$ *for any construction.* This is on account of R.L. Dobrushin [100], where in case of the simple symmetric random walk, as $n \to \infty$, it is concluded that

$$(\xi(1,n) - \xi(0,n))/(2^{1/2}n^{1/4}) \xrightarrow{\mathcal{D}} Z_1|Z_2|^{1/2}, \tag{2.81}$$

while for Brownian local time it can be shown (cf., e.g., M. Yor [305]) that

$$(L(1,n) - L(0,n))/(2n^{1/4}) \xrightarrow{\mathcal{D}} Z_1|Z_2|^{1/2}, \tag{2.82}$$

where Z_1 and Z_2 are independent standard normal random variables.

On the other hand, it follows from Theorem 1.2 of A.N. Borodin [35] that n^ϵ of (2.80) can be replaced by $\log n$ so that (2.80) holds true with $O(n^{1/4}\log n)$. Moreover, Csörgő and Horváth [**109**] conclude that *Révész's Skorohod-type construction is the best possible such construction* by showing that the following rate

$$\sup_x |\xi(x,n) - L(x,n)| = O(n^{1/4}(\log\log n)^{1/4}(\log n)^{1/2}) \quad \text{a.s.} \tag{2.83}$$

is exact for the same construction. Moreover, due to having also (cf. [**69**])

$$\limsup_{n\to\infty} |\xi(1,n) - \xi(0,n)|/(n^{1/4}(\log\log n)^{3/4}) = \left(\frac{128}{27}\right)^{1/4} \quad \text{a.s.,} \tag{2.84}$$

as well as (cf. [**108**])

$$\limsup_{n\to\infty} |L(1,n) - L(0,n)|/(n^{1/4}(\log\log n)^{3/4}) = 2 \cdot \frac{2^{5/4}}{3^{3/4}} \quad \text{a.s.,} \tag{2.85}$$

having

$$\sup_x |\xi(x,n) - L(x,n)| = o(n^{1/4}(\log\log n)^{3/4}) \quad \text{a.s.,} \tag{2.86}$$

is impossible and, therefore, *having* (2.86) *with* $O(n^{1/4}(\log\log n)^{3/4})$ *is best possible for any construction.* Thus, unlike when approximating partial sums by a Wiener process, the latter best possible construction for random walk local time strong approximation is not much better than the best possible Skorohod embedding rate of (2.83). Only the $(\log n)^{1/2}$ term of the latter could be changed, and only to $(\log\log n)^{1/2}$ by any other construction. We are *not aware of any such construction that would achieve this best possible minimal gain.*

The authors of [**109**] also note that the *best possible version of Révész's Skorohod construction as in* (2.83) *remains true if X_1 takes only integer values and the moment generating function of X_1 is finite in a neighbourhood of zero.* Consequently, in the latter case, (2.83) *remains true in the context of* [72] *as well.* For further results along the lines of best possible approximations when X_1 is assumed to have only a finite number of moments, we refer to R.F. Bass and D. Khoshnevisan [16].

Jump-started by Révész 's paper [238] of 1980/81, in addition to the ones already mentioned, there is *a long sequence of further outstanding papers in the eighties, studying various aspects of P. Lévy's local time.*

Paper [**71**] extends the results of [72] that deal with approximating local times of partial sums in the lattice case, to a suitably defined *local time of a more general class of random walks.* The proofs in [**71**] involve *two strong invariance principles* that were proved *via* KMT *construction* in [**60**] *for approximating the occupation time of partial sums* under sufficient moment, as well as under finite moment generating function, conditions *by that of a Wiener process* (cf. (2.79)). Also, in paper [**71**] the quest of proceeding from the just mentioned strong approximations of occupation times to those of local times is facilitated by introducing a *discrete version of the Tanaka formula.*

Paper [**60**] is revisited by Révész [243] in [V2]. One of the main aims of his paper is to address the problem of estimating the Brownian local time process $L(x,t)$ of (2.79) uniformly in x on the real line and $0 \leq t \leq n$, as $n \to \infty$, given only the observations $W(1), W(2), \ldots, W(n)$. Révész shows that a conjecture in [**60**] concerning one of the there proposed estimators of $L(\cdot,\cdot)$ is not true. He then insightfully introduces a new, unbiased estimator that looks more natural than the ones proposed and studied in [**60**]. In [**60**] the authors also obtain analogous results for estimators of $L(x,t)$, $0 \leq t \leq 1$, via observations of $W(t)$ at $t = i/n$, $i = 1, 2, \ldots, n$.

A celebrated result of P. Lévy states (cf., e.g., F.B. Knight ([179], Theorem 5.3.7)) that

$$\{Y(t), M(t); t \geq 0\} \stackrel{\mathcal{D}}{=} \{|W(t)|, L(0,t); t \geq 0\}, \tag{2.87}$$

where $M(t) := \sup_{0 \leq s \leq t} W(s)$, $Y(t) := M(t) - W(t)$, $W(\cdot)$ is a standard Wiener process, and $L(0,t)$ is its local time at zero.

The result of paper [**58**] on *large increments of the local time* $L(\cdot,\cdot)$ *of Brownian motion* are analogous to those of [**43**] for a Wiener process that are quoted in part in Section 2.6. In view of (2.87), in [**58**] the authors study *large fluctuations of* $L(0,t)$ *via those of* $M(t)$ in t. A similar study of $L(x,t)$ in t uniformly in x yields slightly different results. Hand in hand with those of [**43**], these two sets of results have ever since played an important role in studying path properties of various local times and additive functionals.

For local time analogues of the "how small" topics of [**44**] we refer to E. Csáki and A. Földes [66], and for a review of further related results as well, to Sections 11.1–11.4 of [242], where *some Strassen type theorems* of Csáki and Révész [73] are also discussed in this context in combination with those of M.D. Donsker and S.R.S. Varadhan [103].

Inspired by [**43**], paper [**58**] *with* Csáki, Földes, Révész *is the first joint work* of Miklós Csörgő *with* Endre Csáki and Antónia Földes. For a most insightful review of this continued collaboration, plus some more on *strong approximation of local time and additive functionals, path properties of Cauchy principal values of Brownian local time, iterated processes, level crossings of the empirical process, Vervaat and Vervaat error processes* and *Banach space valued stochastic processes*, we refer to [69] the first paper in [V2] titled "Our joint work with Miklós Csörgő ", by Endre Csáki, Antónia Földes and Zhan Shi, with 85 references that include [A1], [A2], [**43**], [**44**], [**58**], [**102**], [**108**], [**115**], [**119**], [**125**], [**130**], [**133**], [**134**], [**150**], [**152**], [**153**], [**156**], [**164**], [**167**], [**195**], [**170**], [**172**], [**176**], [**178**], [**182**], [**183**], [**185**] and [**187**]. This beautiful exposition much facilitates our own review of Miklós Csörgő's *contributions to studying fine analytic path properties of Gaussian and related stochastic processes.* In particular, for details on the just mentioned papers and many other related ones that are quoted in there, we conveniently refer to [69], published in [V2]. Most of these works are also discussed in [**179**] and [**186**], the latter in [B3].

In view of [69] in [V2] that takes off from [**43**], [**44**], and surveys joint work with Miklós Csörgő from 1983 to 2002 *on path properties of stochastic processes*, we list here the papers [**34**], [**38**], [**39**], [**50**], [**51**], [**52**], [**60**], [**69**], [**77**], [**78**], [**95**], [**103**], [**114**], [**122**], [**124**], [**127**], [**131**], [**136**], [**138**], [**139**], [**140**], [**144**], [**145**], [**151**], [**165**], [**180**] and [**181**] that, globally speaking, can also be classified as *works on studying fine analytic properties of the path behaviour of stochastic processes*. Though the expositions [**34**], [**38**], [**39**], appeared earlier than [**43**] and [**44**], the former three are follow-up investigations to [**43**]. Paper [**44**] (cf. (2.75)–(2.76) in this essay) inspired [**50**] with Révész, where the authors conclude that the *Wiener sheet is nowhere differentiable in any direction in the open unit square*. Naturally, mutatis mutandis, the same holds true in the case of higher time parameter Wiener random fields as well. For *nondifferentiability of curves in general on the Brownian sheet*, we refer to R.C. Dalang and T. Mountford [83] and to Section 6 of Khoshnevisan [171] in [V2]. The papers [**102**], [**103**], [**114**], [**115**] with Zheng-yan Lin, together with B. Schmuland [250], [251], and I. Iscoe and D. McDonald [153], [154] initiate the study of the path properties of the *infinite dimensional Ornstein-Uhlenbeck processes* that were introduced by Donald A. Dawson [87] (cf. (3.62) and (3.63)). For a review of further developments we refer to the books [188], [189] and the references therein, as well as to Section 5 of Csáki et al. [69] in [V2].

"Mesure du voisinage", and long Brownian and random walk excursions.We continue with quoting from [**186**] by Miklós Csörgő in [B3].

"The *definition of the local time* of a standard Wiener process $W(\cdot)$ (cf. (2.79)) *is extrinsic* in the sense that, given the random set with $t > 0$ fixed,

$$A_t := \{s : 0 \le s \le t, W(s) \in A\} \tag{2.88}$$

of the occupation time $H(A, t)$ of $W(\cdot)$ as in (2.79), one cannot recover the local time $L(x, t)$ for any $x \in A \subset \mathrm{R}^1$ via this definition.

Seeking *an intrinsic definition*, P. Lévy ([187], p. 226) proposed the following approach to this problem: Let $N(h, x, t)$ be the number of excursions of $W(\cdot)$ away from x that are greater than h in length and are completed by time t. Then the "*mesure du voisinage*" of $W(\cdot)$ at time t is defined to be $\lim_{h\downarrow 0} h^{1/2}N(h, x, t)$, which is shown by Lévy (cf. K. Itô and H.P. McKean, Jr. ([156], p. 43)) that for all $x \in \mathrm{R}^1$ and all positive t one has

$$\lim_{h\downarrow 0} h^{1/2}N(h, x, t) = \sqrt{\frac{2}{\pi}}L(x, t) \quad \text{a.s.} \tag{2.89}$$

Note that the *"mesure du voisinage"*, say at $x = 0$, can be computed in terms of the zeros of $W(t)$ alone. Hence, this *gives an intrinsic meaning to* $L(0, t)$*, as well as to* $L(x, t)$ via (2.89) for any $x \in \mathrm{R}^1$.

Ed Perkins [212] in 1981 showed that the exceptional null sets, which may depend on x, can be combined into a single null set off which the above convergence is uniform in x. Paper [**77**] with Révész in 1986 re-establishes Perkins' uniform in x version of Lévy's result and concludes also the following rate of convergence estimate for any fixed t

$$\sup_{x\in\mathrm{R}^1}\left|\left(\pi\frac{h}{2}\right)^{1/2} N(h, x, t) - L(x, t)\right| = o\left(h^{1/4}\log\frac{1}{h}\right) \quad \text{a.s.} \tag{2.90}$$

as $h \downarrow 0$, as well as a similar one in the case when a Wiener process is observed for a long time t and the number of long (but much shorter than t) excursions is considered. For a review of these and some related results we refer to Sections 13.2 and 13.3 of book [242].

In paper [**78**] with Révész and Horváth in 1986 it is shown that the rate of convergence in (2.90) is nearly best possible even for a fixed level. Looking at some further developments now in this regard, it follows from Theorem 1 of L. Horváth [148] that *for a fixed level* $x \in \mathrm{R}^1$ the best possible rate of convergence in (2.90) is $O(h^{1/4}(\log\log(1/h))^{1/2})$ with the *exact "constant"* $(2\pi)^{1/4}(L(x,t))^{1/2}$. Furthermore, without being able to identify the exact constant, it is shown by D. Khoshnevisan [169] that the best possible rate of convergence in (2.90) *in the case of* $\sup_{x\in\mathrm{R}^1}$ is $O(h^{1/4}(\log(1/h))^{1/2})$. We note in passing that the result in (2.90), as well as that of the latter best possible version by Khoshnevisan, hold uniformly in $t \in T$ as well, where T is an arbitrary nonrandom compact subset of $[0,\infty)$.

In addition to the just mentioned best possible improvement of (2.90), D. Khoshnevisan [169] also establishes similar *best possible rates of convergence*, with specified exact constants, for the uniform and local approximations of Brownian local times by *P. Lévy's occupation time, and for* his so-called *downcrossing theorem.*

Concerning the already hinted at analogue of (2.90) for long excursions in a long time, in[**77**] the following result is proved: *For some* $0 < \alpha < 1$ *let* $0 < a_t < t^\alpha$ *be a nondecreasing function of* t *so that* a_t/t *is non-increasing. Then, as* $t \to \infty$,

$$\sup_{x\in\mathrm{R}^1}\left|\sqrt{\pi\frac{a_t}{2}}N(a_t,x,t) - L(x,t)\right| = o\left((ta_t)^{1/4}\log\frac{t}{a_t}\right) \quad \text{a.s.} \tag{2.91}$$

For a *random walk analogue of this result* with a similar rate of convergence, as well as for further related ones along these lines, we refer to Section 13.2 of [242] and, borrowing a bit from the nineties, to paper [**131**] with Révész (cf. (2.92) below). The problem of best possible exact rates of convergence in this context, including that of (2.91) and its local version, appear to be still open."

The above hinted at *random walk analogue of* (2.91) reads as follows. Let $S(k)$ be a simple symmetric random walk with local time $\xi(x,n)$ as in (2.78). Let $M(a,x,n)$ be the number of excursions of duration greater than a of $S(k)$, $k = 0,1,\ldots$, away from $x \in \mathbb{Z}^1$ that are completed by time n. Then, as $n \to \infty$,

$$\sup_{x\in\mathbb{Z}^1}\left|\sqrt{\frac{\pi a_n}{2}}M(a_n,x,n) - \xi(x,n)\right| = o((na_n)^{1/4}\log(n/a_n)) \quad \text{a.s.,} \tag{2.92}$$

where a_n is a sequence of integers such that $n^{-1/5}a_n \to \infty$ and $a_n \le n^\alpha$ for some $\alpha \in (0,3/5]$.

When discussing above the optimal nature of Révész's Skorohod-type construction in [238], we made references to, and cited as well, some fundamental results in (2.81) and (2.82) from R.L. Dobrushin [100] and M. Yor [305], respectively, and (2.84), (2.85) from [**69**], **108**], respectively. In this regard we note that the famous theorem of R.L. Dobrushin [100] that is quoted in (2.81) and an extension of his method of proof have, for example, led to the result that is quoted in (2.84) from paper [**69**], as well as to further results in there, on studying the intuitive notion that $\xi(x,n)$ is close to $\xi(y,n)$ if x is close to y, i.e., the *stability* of the *local time* of a symmetric random walk (cf. (3.48) and (3.49) in this exposition). The results in [**69**] along these lines were improved and also extended to Brownian local time by E. Csáki and A. Földes [67], [68]. In particular, in their 1988 paper [68] they give a direct proof of the result that is quoted from paper [**108**] in (2.85).

Strong approximations of increments from zero of Brownian local time and random walk additive functionals by iterated processes. The influential paper [**108**] with

E. Csáki, A. Földes and P. Révész in 1989 establishes a *strong embedding theorem* for the local time increment $L(x,t) - L(0,t)$ of a Wiener process *into* $2W(x, \hat{L}(0,t))$, $x \geq 0$, where $\hat{L}(0,t)$ is the local time of Wiener process that is independent of the Brownian sheet $W(x,t)$ and it is also near enough to $L(0,t)$. For details we refer to Section 2.2 of [69] in [V2].

In a similar vein, in [**134**] the authors establish *strong approximations of additive functionals* of a sequence of partial sums $\{S_i\}_{i=0}^{\infty}$ of i.i.d. integer valued random variables. Namely, *à la* dealing with $L(x,t) - L(x,0)$ in [**108**], in [**134**] they study

$$\sum_{i=1}^{n} f(S_i) - \bar{f}\xi(0,n) \quad \text{via constructing} \quad W^{(2)}\left(\frac{1}{\sigma^2}L^{(1)}(0,n\sigma^2)\right) \tag{2.93}$$

to be a.s. near enough to it, where $f(x)$, $x \in \mathbb{Z}^1$, is such that $\bar{f} := \sum_{x=-\infty}^{\infty} |f(x)| < \infty$, $\xi(0,n)$ is the local time of S_i at zero up to time n, $\sigma^2 = EX_1^2$, $W^{(1)}(\cdot)$ and $W^{(2)}(\cdot)$ are two independent standard Wiener processes, $L^{(1)}(0,\cdot)$ is local time of $W^{(1)}(\cdot)$, and $\sigma^2\xi(0,n)$ and $L^{(1)}(0,n\sigma^2)$ are constructed to be a.s. near enough to each other as well. For details we refer to Section 2.3 of [69] in [V2]. Some of the consequences in case of a simple symmetric random walk read as follows:

$$\xi(1,n) - \frac{\xi(0,n)}{\sqrt{2\xi(0,n)}} \xrightarrow{\mathcal{D}} Z_1, \quad \text{as } n \to \infty, \tag{2.94}$$

$$n^{-1/2}(\xi(0,n)) \xrightarrow{\mathcal{D}} |Z_2|, \quad \text{as } n \to \infty, \tag{2.95}$$

and a statement that is identical to that of Dobrushin's theorem in (2.81), where Z_1 and Z_2 are independent standard normal random variables.

The strong law in (2.84) can also be deduced from the more general results of paper [**134**] where, in addition to the already mentioned additive functionals, the authors also deal with a similar *strong approximation of* $\int_0^t g(W(s))ds = \int_{-\infty}^{\infty} g(x)L(x,t)dx$, where $g(\cdot)$ is assumed to be integrable on R^1 and $W(\cdot)$ is standard Brownian motion. We note also that the result in (2.85), which is one of the strong consequences of the strong invariance principle of paper [**108**], can also be deduced from results of [**134**] concerning strong approximations of $\int_0^t g(W(s))ds = \int_{-\infty}^{\infty} g(x)L(x,t)dx$.

As evidenced by these discussions of the local time milieu after Révész's paper [238] of 1980/81, the interplay of, and the progress made by, the papers [**69**], E. Csáki and A. Földes [67], [68], [**108**] and [**134**] constitute impressive developments in studying local times and additive functionals. They also initiate the more recent investigations on the old problem of *finding rates of convergence in the ratio ergodic theorem*. Paper [**134**] is already a step in this direction as well, followed by E. Csáki and M. Csörgő [**152**] on *additive functionals of Markov chains*, and more recently expounded on by Xia Chen [50]. For further developments along these and many other related lines we refer to [242], the papers [**187**], [70] and [74] in [B3], and to the book of Davar Khoshnevisan [170].

Testing for independence. *Back to empirical processes*, paper [**42**] initiates a *study* of *the Hoeffding, Blum, Kiefer, Rosenblatt* multivariate empirical process via strong approximations. Let $F_n(x)$, $x = (x_1, \ldots, x_d) \in \mathrm{R}^d$, be the empirical distribution function based on a random sample of random d–vectors ($d \geq 2$) of a continuous distribution function F, and denote by $F_{ni}(x_i)$, $i = 1, \ldots, d$, the corresponding marginal empirical distribution functions. Based on the process

$$T_n(x) := F_n(x) - \prod_{i=1}^{d} F_{ni}(x_i), \quad d \geq 2, \tag{2.96}$$

W. Hoeffding [146], when $d = 2$, and J.R. Blum, J. Kiefer and M. Rosenblatt [30], when $d \geq 2$, constructed distribution-free tests of independence. In [**42**] strong invariance principles are established by approximating $T_n(x)$ by appropriate Gaussian processes. Papers [**47**], [**55**] report on further developments along these lines, culminating in the papers [**56**] and **66**] with Derek S. Cotterill.

As of [**56**], let $V_{n,d}(x)$, $d \geq 1$, $x \in \mathrm{R}^1$, be the distribution function of the classical *Cramér-von Mises statistic* based on n independent d–dimensional random vectors distributed uniformly on the unit cube of R^d, and let $V_d(x)$ be the limiting distribution function of $V_{n,d}(x)$ as $n \to \infty$. The authors deduce from basic results of F. Götze [133] that

$$\Delta_{n,d} := \sup\{x \in \mathbf{R}^1 : |V_{n,d}(x) - V_d(x)|\} = O(n^{-1}) \ \text{ for any } \ d \geq 1. \tag{2.97}$$

Recursive formulae are given for all the moments and cumulants corresponding to the limiting V_d. Using the Cornish-Fisher asymptotic expansion, based on six cumulants, the authors compile tables of the critical values of V_d, corresponding to the usual testing levels. These tables run from $d = 2$ to 50. Such tables were previously known only for $d = 1, 2, 3$. An interesting finding is that $K_{k,d} = O(e^{-d})$, as $d \to \infty$, for the k^{th} cumulant $K_{k,d}$. Consequently, the tables presented become more and more precise as the dimension grows (cf. p. 239 of [**56**]).

Studying the Hoeffding [146], Blum, Kiefer and Rosenblatt [30] *independence criterion*, paper [**66**] builds on results of the latter papers and those of [**42**]. Let $T_n(x)$ be as in (2.96), and define the Gaussian process $T(\cdot)$ by

$$\{T(y); y \in I^d, d \geq 1\} := \left\{ B(y) - \sum_{i=1}^{d} B(\mathbf{1}, y_i, \mathbf{1}) \prod_{j \neq i} y_j; y = (y_1, \ldots, y_d) \in I^d, d \geq 2 \right\}, \tag{2.98}$$

where $\{B(y);\ y \in I^d,\ d \geq 1\}$ is a Brownian bridge over the d–dimensional unit cube I^d. With T_n and T as in (2.96) and (2.98) respectively, let $\Gamma_{n,d}$ be the distribution function of the random variable $C_{n,d} := \int_{\mathrm{R}^d} nT_n^2(x) \prod_{i=1}^d dF_i(x_i)$, where $F_i(x_i)$ is the i^{th} marginal of the underlying continuous distribution function F, and let Γ_d be the distribution function of the random variable $C_d := \int_{I^d} T^2(y) dy$. Blum, Kiefer, Rosenblatt [30] obtained the characteristic function of the random variable C_2 and tabulated critical values for Γ_2. With the help of their representation of the Gaussian process $T(\cdot)$ as in (2.98), the authors in [**66**] find the characteristic function of the random variable C_d for $d \geq 2$, give details as to how to calculate critical values of Γ_d for all $d \geq 2$, and provide tables for the usual testing levels of significance for $d = 2$ to 20. With T_n, F_n and F_{ni} as in (2.96), the latter are also shown to be approximate critical values for the statistics $\hat{C}_{n,d} := \int_{\mathrm{R}^d} nT_n^2(x) dF_n(x)$ and $\tilde{C}_{n,d} := \int_{R^d} nT_n^2(x) \prod_{i=1}^d dF_{ni}(x)$, whose large values can be used as critical regions for $H_0 : F \in \mathcal{F}_0$, where $\mathcal{F}_0$ is the class of all those continuous distribution functions which are products of their associated one-dimensional marginal distribution functions. Put $\triangledown_{n,d} = \sup_{0 \leq x < \infty} |\Gamma_{n,d}(x) - \Gamma_d(x)|$, $d \geq 2$. Using their characteristic function of C_d and the invariance results of [**42**] for the process $T_n(\cdot)$ as in (2.96), the authors in [**66**] also conclude rates of convergence for the latter distance as follows (cf. also Tusnády [289], and (2.10) in this exposition):

$$\triangledown_{n,d} = \begin{cases} O(n^{-1/2} \log^2 n) & \text{if } d = 2, \\ O(n^{-1/2(d+1)} (\log n)^{3/2} & \text{if } d \geq 3. \end{cases} \tag{2.99}$$

In view of (2.97) it is reasonable to believe that in (2.99) one should have $\bigtriangledown_{n,d} = O(n^{-1})$ for any $d \geq 2$. For comments on this problem we refer to page 10 of [**66**]. A summary of the papers [**42**], **47**], [**55**] [**56**], [**66**] can be found in [**68**].

On the 1981 NSF-CBMS Regional Conference on Quantile processes. The *quantile papers* [**29**], **40**] and book [A1] with Pál Révész played a seminal role in the ten lectures given by Miklós Csörgő as *Principal Lecturer* of the National Science Foundation (NSF)–Conference Board of the Mathematical Sciences (CBMS) Regional Conference on Quantile Processes at Texas A & M University in 1981, which was foresightedly organized by Emanuel Parzen. The already frequently mentioned 1983 SIAM monograph [A2] resulted from these ten lectures, while featuring also some of the topics of [**29**], [**30**], [**40**], [A1], [**41**], [**42**], [**49**], [**54**], as well as those of the preliminary versions of [**57**], [**59**], [**61**], [**62**], [**64**], [**65**], [**67**], [**70**], [**73**], [**76**] and [A3]. As regards the latter papers, we note that *papers* [**64**] *and* [**67**] *also signal the beginning of* Miklós Csörgő's *collaboration with* Lajos Horváth. Paper [**64**] has already been mentioned on occasions when dealing with the landmark quantile papers [**29**], [**40**] in some detail in Section 2. As to [**67**] and its relationship to [A2], we are to say a few words now.

Quantiles under random censorship. The *sample distribution* function *for censored biostatistical observations* is defined by the Kaplan-Meier *product-limit* (PL) *estimator* (cf. Kaplan and Meier [160]). There is an immense literature dealing with this estimator, and it is widely used in applications. On the other hand, Parzen [211] in [V2] notes that, in his opinion, the *sample quantile* (inverse distribution) *of* the Kaplan-Meier *estimator* is a neglected diagnostic tool. This indeed appears to be true, though the theoretical foundations for using this diagnostic tool are also well established. In particular, Chapter 8 of [A2] that is based on two Carleton University technical reports in 1982, one by E.-E. Aly and M. Csörgő and the other by L. Horváth, studies the problem of strong approximation of the *uniform* PL–*quantile process* via an appropriately normalized *generalized* Kiefer *process*, as well as the *distance* of *the latter uniform* PL–*quantile process* from *the corresponding general* PL–*quantile process* under the conditions (i), (ii), (iii) of (2.47) and (iv), (v) of (2.54). Improved versions of the results of Chapter 8 of [A2] can be found in paper [**67**]. Moreover, [**116**] with C.-J.F. Chung and L. Horváth deals with constructing *confidence bands for quantile function under random censorship* (cf. also [55]).

Empirical reliability and concentration processes. We quote from the PREFACE of the 1986 Springer-Verlag monograph [A3] with Sándor Csörgő and Lajos Horváth:

> Miklós Csörgő and David M. Mason initiated their collaboration on the topics of this book while attending the CBMS-NSF Regional Conference at Texas A & M University in 1981. Independently of them, Sándor Csörgő and Lajos Horváth have begun their work on this subject at Szeged University. The idea of writing a monograph together was born when the four of us met in the Conference on Limit Theorems in Probability and Statistics, Veszprém 1982. This collaboration resulted in No. 2 of Technical Report Series of the Laboratory for Research in Statistics and Probability of Carleton University and University of Ottawa, 1983. Afterwards David M. Mason has decided to withdraw from this project. The authors wish to thank him for his contributions. In particular, he has called our attention to the reverse martingale property of the empirical process together with the associated Birnbaum-Marshall inequality (cf. the proofs of Lemmas 2.4 and 3.2) and to the Hardy inequality (cf. the proof of part (iv) of Theorem 4.1). These and several other related remarks helped

us push down the moment conditions to $EX^2 < \infty$ in all our weak approximation theorems.

Reviewing this book in Short Book Reviews ISI, Charles M. Goldie of University of Sussex writes:

> Readership: Statistician, reliability theorist, economist, biometrician
>
> The 'total time on test' is used in reliability engineering, the 'Lorenze curve' in economics, and 'mean residual life' in biostatistics. Here is a unified treatment of the asymptotics of all these, based on strong approximation of stochastic processes and the authors' powerful quantile methods. Processes relating to the above statistics involve sums of order statistics in some form, and other such processes are introduced for specific purposes. Approximating processes are Gaussian. The treatment is clear and thorough and quite concrete, and the asymptotics of many specific functionals are worked out. Estimation procedures including bootstrap methods are also considered.

3 From the Mid-eighties on into the Twenty-first Century

Weighted approximations of empirical and quantile processes. The 1983 SIAM monograph [A2], together with the papers [**29**], [**40**], [**64**] and the 1986 Springer-Verlag monograph [A3] on *empirical reliability* have led to the collaboration of Mikós Csörgő, Sándor Csörgő, Lajos Horváth and David M. Mason [CsCsHM] on the fundamental paper [**74**] on *weighted approximations of empirical and quantile processes.* The CsCsHM [**74**] proof of these approximations in weighted sup-norm metrics is based on a refinement of the CsR [**40**] inequality as quoted in (2.27), which in turn is based on the KMT [181] inequality as cited in (2.19). The desired modification of the CsR [**40**] inequality (2.27) in CsCsHM [**74**] is to "pick up" the tail behaviour of the approximation via modifying the construction, and it reads as follows: *There exists a probability space with independent uniform–*$[0,1]$ *random variables* $U_1, U_2, \ldots,$ *and a sequence of Brownian bridges* $B_1, B_2, \ldots,$ *such that for all* $n \geq 1$, $1 \leq d \leq n$ *and* $x > 0$

$$P\Big\{ \sup_{0 \leq y \leq d/n} |u_n(y) - B_n(y)| > n^{-1/2}(a_1 \log d + x) \Big\} \leq b_1 \exp(-c_1 x) \tag{3.1}$$

and

$$P\Big\{ \sup_{1-d/n \leq y \leq 1} |u_n(y) - B_n(y)| > n^{-1/2}(a_2 \log d + x) \Big\} \leq b_2 \exp(-c_2 x), \tag{3.2}$$

where a_i, b_i, c_i $(i = 1, 2)$ *are suitable positive constants.*

For details of proof we refer to [**74**] and Section 3.2 of Csörgő and Horváth [CsH] [A4]. With $d = n$ the inequalities (3.1) and (3.2) coincide and, on the CsCsHM [**74**] probability space, yield (2.20) with the Brownian bridge sequence $\{B_n\}$ as in (3.1) $\equiv$ (3.2). Due to (2.23), this particular sequence of Brownian bridges approximates the uniform empirical process α_n (cf. (2.13)) at the Kiefer [174] rate of convergence in (2.23). In view of this, inequalities (3.1) and (3.2) lead to the following important weighted approximations for α_n and u_n (cf. (2.13) and (2.14) respectively): *on the* CsCsHM [**74**] *probability space, as* $n \to \infty$ *we have, for any* $0 \leq \nu_1 < 1/2$,

$$\sup_{1/(n+1) \leq y \leq n/(n+1)} |u_n(y) - B_n(y)|/(y(1-y))^{1/2-\nu_1} = O_P(n^{-\nu_1}) \tag{3.3}$$

and for any $0 \le \nu_2 < 1/4$,

$$\sup_{0\le y\le 1} |\alpha_n(y) - \bar{B}_n(y)|/(y(1-y))^{1/2-\nu_2} = O_P(n^{-\nu_2}) \tag{3.4}$$

where $\bar{B}_n(y) = -B_n(y)$, $1/n \le y \le 1 - 1/n$, $n \ge 2$, *and zero elsewhere.*

The result of (3.3) is Theorem 2.1 of CsCsHM [**74**], while that of (3.4) is Corollary 4.2.2 of the same paper. The complete proofs of the inequalities (3.1), (3.2) entail a substantial amount of technical detail (cf. [**74**] and CsH [A4]). CsH [**80**] and, independently, D.M. Mason [197] gave mathematically equivalent short and simple proofs of (3.3) based directly on the KMT [181] strong approximation of partial sums as quoted in (2.19). Both papers noted also that the KMT [181] approximation in their proof of (3.3) can be substituted by the Skorohod [267] embedding, resulting in having (3.3) for all $0 \le \nu_1 < 1/4$ instead of for all $0 \le \nu_1 < 1/2$. CsH [**80**] also gave a short proof for (3.4). Noting that, for each $n \ge 1$,

$$\{E_n(t),\, 0 \le t < 1\} \stackrel{\mathcal{D}}{=} \{n^{-1}N(tS_{n+1}),\, 0 \le t < 1\}, \tag{3.5}$$

where $N(x) = \sum_{i=1}^{\infty} 1\!\!1_{[0,x]}(S_i)$, with S_i as in (2.21), is a Poisson process, their proof of (3.4) is based on this Poissonization and the CsR [**43**] strong laws for the increments of a Wiener process (quoted here in Section 2.6) used in conjunction with a Wiener process strong approximation to the Poisson process.

In the Addendum of his paper [197] Mason also presents a short proof for (3.4), constructed in the spirit of its original proof in CsCsHM [**74**]. The main tool in Mason's short proof of (3.4) is an inequality for the oscillation modulus of the uniform empirical process.

CsCsHM [**74**] also note (cf. their Remark 2.1) that an analogous theory of their weighted approximations is also feasible via the KMT [181] inequality (2.15). Just as the key results in the CsCsHM [**74**] theory are the inequalities (3.1), (3.2), refinements of the CsR [**40**] inequality (2.27), the key to the alternative theory need be similar refinements of the KMT [181] inequality (2.15). D.M. Mason and W.R. van Zwet (*Ann. Probab.* **15** (1987), 871–884) succeeded in doing this, and thus have constructed a probability space dual to that of CsCsHM [**74**] as in (3.1) and (3.2) on which (3.3) holds for all $0 \le \nu_1 < 1/4$ and (3.4) holds for all $0 \le \nu_2 < 1/2$, i.e., reversing the bounds on ν_1 and ν_2 in view of the Kiefer [174] result (2.23).

Extensions of the classical Erdős–Rényi–Kolmogorov–Petrovski tests for upper and lower class functions of Brownian motion. Just like the papers that we have discussed so far on strong and weak approximations of empirical, quantile and partial sums processes have led to studying path properties of the approximating Gaussian processes, paper [**74**] has played a similar role as well (cf. also [**83**]). In particular, when determining the optimal class of weight functions for their weighted approximations, CsCsHM [**74**] build their criteria (cf. Lemmas 3.1, 3.2, and Remarks 5.1, 5.2 in [**192**] in [V2]) on variants of the classical Erdős–Feller–Kolmogorov–Petrovski [EFKP] tests for upper and lower class functions of Brownian motion (cf. Itô and McKean [156], Petrovski [214], Erdős [115], Feller [119], [120]). Moreover, proof of CONJECTURE of [**74**] in [**127**] with Q.-M. Shao, B. Szyszkowicz has led to an extension of integral criteria for Khinchine's local, as well as global, LIL for Brownian motion. Consequently, the authors of [**127**] conjectured that similar extensions of the classical EFKP tests for upper and lower class functions of Brownian motion should be also possible. This, in turn, was proved to be true by Keprta [165], [166]. For details on these matters we refer to Section 5.1 of [**189**].

Theory of weighted approximations extending that of weak convergence in weighted sup-norm metrics. The weighted approximations (3.3), (3.4) have found numerous and wide ranging applications in probability theory and statistics. To begin with, they provide simple proofs for the important Rényi [224] (cf. also [**9**] and Csáki [60], [63]), Chibisov [52], O'Reilly [205], Eicker [108] and Jaeschke [157] theorems on the weak convergence and limiting distribution of weighted uniform empirical and quantile processes. Moreover, the (3.3) and (3.4) based *theory of weighted approximations goes beyond that of weak convergence in weighted sup-norm metrics* in that it also yields examples of functionals in sup-norm converging in distribution (cf., e.g., Theorem 4.2.3 of CsCsHM [**74**]) when the distribution law of the limiting random function is not necessarily a Radon measure. This type of convergence in distribution cannot be argued via the usual approach of demonstrating convergence of finite dimensional distributions in combination with proving tightness with respect to a weighted sup-norm topology. For further such examples we refer to [**171**] with R. Norvaiša and B. Szyszkowicz and, in the context of the domain of attraction of the normal law, to Corollary 3.4, Remark 3.7, Corollary 5.2 and Remark 5.2 of [**192**] in [V2].

Concerning invariance principles as in [**171**] with Norvaiša and Szyszkowicz, let B_w be a *non-separable Banach space of real valued functions endowed with a weighted sup-norm.* In [**171**] the authors consider partial sum processes as random functions with values in B_w. Their main result deals with *convergence of distributions* of certain functionals in the case *when* the *Wiener measure is not necessarily a Radon measure on* B_w, and hence the usual approach of proving tightness with respect to a weighted sup–norm topology is not applicable. Just like in CsH [**105**] and Szyszkowicz [278], [280], this difficulty is overcome by establishing this kind of convergence in distribution via strong approximation methods. Moreover, in [**171**], the corresponding results of CsH [**105**] and Szyszkowicz [278], [280] are extended to arbitrary weight functions. It is also shown in [**171**] that the above notion of *convergence in* B_w *to a not necessarily Radon measure is equivalent to having a bounded central limit theorem.*

In view of the phenomena of functionals in sup-norm converging in distribution when the distribution law of the limiting random function is not necessarily a Radon measure in weighted sup-norm topology, it became evident that, in addition to weak convergence of random functions with values in metric spaces, convergence in distribution of their functionals in terms of various weighted metrics should be studied on their own for the sake of applications and for gaining further insight into these matters. The papers [**100**], [**117**] with Horváth deal with *weighted L_p functionals of empirical and quantile processes.* Extending the studies of [**100**], in [**117**] the authors establish limit theorems for the distributions in weighted L_p norms of the quantile processes γ_n and ρ_n as in (2.35) and (2.44) respectively. Under appropriate conditions, the limiting random variables are represented as integrals of weighted Wiener and exponential partial sums processes. In [**135**] with Horváth and Q.-M. Shao the authors find a *necessary and sufficient condition for the weak convergence of the uniform empirical and quantile processes to a Brownian bridge in weighted L_p–distances.* Unlike in the case of weighted sup-norm topologies as in [**74**], in this context the corresponding weighted L_p–functionals of these processes are shown to converge in distribution under the same condition to the corresponding functionals of a Brownian bridge. The same is true for partial sums processes in this regard (cf. Szyszkowicz [279]). The proofs are based on *dichotomy theorems for integrals of stochastic processes* (cf., e.g., Lemma 4.1 in [**192**] in [V2]). For a review of weighted approximations in probability of partial sums and empirical processes we refer to [**137**]. For a view of, and necessary and sufficient conditions for, *invariance principles in probability for empirical and partial sums processes with sample paths in Banach* *function spaces*, we refer to [**146**] and [**154**] with Rimas Norvaiša. In [**166**] with Norvaiša, the convergence in distribution of *standardized sequential empirical processes* is related to the *extreme value distribution* of a *two-parameter Ornstein-Uhlenbeck process*.

A *weighted invariance principle* is described in [**194**] with Norvaiša *for non-separable Banach space-valued functions* via asymptotic behaviour of a weighted Wiener process. It is proved that, unlike the usual weak invariance principle (cf. Theorem 2.3 of [**171**]), *this weighted variant cannot be characterized via validity of a central limit theorem in a Banach space*. A strong invariance principle is introduced in this context and used to prove the desired weighted weak invariance principle. The result then is applied to empirical processes.

On the impact of weighted approximations. Fashioned after [**135**], paper [**161**] with Horváth and Q.-M. Shao studies *almost sure weighted summability of partial sums of independent random variables*. Some of these results are also reviewed in [**137**]. While at summability, we mention CsH [**96**], where a *Marcinkiewicz-Zygmund type strong law* of large numbers is proved *for random walk summation*. This summation method was introduced by N.H. Bingham (cf. References in [**96**]), who with Makoto Maejima [28] proved respective analogues of the Kolmogorov law of large numbers and the law of the iterated logarithm.

Further to empiricals and quantiles, for strong limit theorems for weighted quantile processes we refer to J.H.J. Einmahl and D.M. Mason [109]. The paper CsH [**99**] investigates the weak limits in suitable function spaces of the *à la* Gnedenko [134] normalized quantile function of the empirical distribution function on the line, and the weak limit of the consequentially normalized empirical distribution function with a transformed argument. The first one of these results can be interpreted as a *functional version of the classical extreme value trio* of B.V. Gnedenko [134].

Paper [**74**] has also led to CsH [**107**] *solving the long standing problem* of proving *central limit theorems for kernel estimators of smooth density functions* in L_p, $1 \le p < \infty$. This was also achieved under random censorship in [**126**] with Edit Gombay and Horváth. The results of [**107**] are also presented in CsH [**98**]. For a review of further central limit theorems in functional estimation we refer to Alain Berlinet [23].

Wide ranging further applications of the weighted approximations of paper [**74**] can be found in CsH [**128**], the CsH [A4] book, and the proceedings volumes edited by M. Hahn, D.M. Mason and D. Weiner [140], and B. Szyszkowicz [V1]. *For* a special tribute to [**74**] and *appreciation of the formulations* (3.3), (3.4) we *refer to* Galen R. Shorack [261] in the latter volume [V1]. For further developments along these lines we refer to D.M. Mason [198].

We note in passing that the CsCsHM [**75**] paper, right after, and very much in the spirit of [**74**], establishes general *invariance principles for integral functions of the empirical process*. Based on the latter invariance principles, paper [**75**] initiates the study of the *criteria* for a distribution *to belong to the domain of attraction of the normal and stable laws with index* $0 < \alpha < 2$ in terms of the tail behaviour of the underlying quantile function. This in turn has led to studies on *asymptotic representations of self-normalized sums* in these domains of attraction (cf., e.g., CsH [**97**], S. Csörgő [76]). The *general invariance principles* of [**75**] *for integral functions of the empirical process* have also evolved into an approach to *domains of attraction based on quantiles* and Feller [121] (cf. S. Csörgő, E. Haeusler and D.M. Mason [79]), and led to significant advances in limit theorems for *sums of order statistics* as well (cf., e.g., S. Csörgő, E. Haeusler and D.M. Mason [80]).

The works in M. Csörgő's list of publications that have figured in *the* already frequently mentioned CsH [A4] 1993 *monograph* are books [A1], [A2], [A3], papers [**9**], [**27**], [**28**], [**29**], [**32**], [**40**], [**62**], [**64**], [**72**] of the 19 year period 1967–1985, and [**73**], [**74**], [**76**], [**80**], [**81**], [**84**], [**88**], [**90**], [**92**], [**99**], [**100**], [**104**], [**113**], [**117**], [**120**], [**121**], [**127**], [**128**], [**135**], [**142**] of the 9 year period 1986–1994. Many of these have already been viewed before. Just reading the titles of the latter period of 1986–1994 gives a rough idea of the scope of [A4].

Weighted approximations–change-point analysis: an interplay. In order to illustrate further developments that have become feasible in view of weighted approximations, adapting style of references to this volume, we quote from the **Preface** *of the* 1997 CsH [A5] *monograph*:

> While working on our previous book, *Weighted Approximations in Probability and Statistics*, Wiley 1993, we were tempted to continue with adding further chapters on studying some selected areas of the mathematical and experimental sciences in which the asymptotic behaviour of functionals of various data-based processes in weighted metrics would appear to play a natural role in resolving certain problems of statistical inference. The complex area of change-point analysis of chronologically ordered observations was one of the topics we had had in mind to write a chapter on. However, due to some imminent enough deadlines and the prospect of a mushrooming number of additional pages, we soon gave up the idea of adding further chapters. At the same time we were kindly encouraged by David Kendall that we should rather continue with writing another book instead, on change-point analysis. We are glad to take this opportunity to thank him for his trust and foresighted advice to us, and hope only that we will not have disappointed him with our endeavour to do just that.
>
> The Kendall and Kendall paper [163] played a seminal role in triggering our interest in studying change-point problems in general, and via weighted approximations (cf. Csörgő and Horváth [**85**]) in particular. Indeed, the latter approach, that in this book is mainly based on some fundamental results of the Hungarian construction school, constitutes the essential backbone of many, if not most, of the proofs. We build directly on our 1993 book, *Weighted Approximations in probability and Statistics*, while aiming at a fairly thorough survey and development of parametric and nonparametric methods for change- point problems.

The CsH [**85**] paper is a first excursion into *change-point analysis*. It has, in particular, also led to an extensive study of *Kendall–Kendall pontograms*. For details and further references, we refer to Section 2.5 and pp. 195-6 of CsH [A4], Szyszkowicz [277], and to the collection of extended abstracts in [B2]. The paper CsH [**129**] has, in turn, led to studying two-time parameter pontograms in [**147**] with Szyszkowicz.

In connection with studying change-point problems in general, in addition to the CsH [A5] monograph, we mention again the collection of extended abstracts of [B2] that is also concerned with *empirical reliability* (cf. also the papers [309], [311], and their references, in [V1]). Questions concerning a possible change in the mean of chronologically ordered observations posed the desirability of *weighted approximations for partial sums processes* in a most natural way, under optimal moment and weight function conditions (cf. [**105**], [278], [280] and [171]). Similarly, a possible change in the distribution of such data naturally led to considering *sequential empirical processes* that are to be weighted *via weight functions of their sequential time parameter* in a best possible way (cf. [**101**], [**147**], [**165**], [281]).

The works in M. Csörgő's list of publications that are made use of in the CsH [A5] *monograph* are books [A1], A2], [A3], [A4], papers [**11**], [**22**], [**29**], [**40**], [**42**], [**43**] of the 12 year period 1968–

1979, and [**74**], [**85**], [**92**], [**93**], [**94**], [**101**], [**105**], [**106**], [**127**], [**135**], [**147**], [**148**], [**155**], [**165**], [**171**] of the 14 year period 1986–1999. The doctoral theses [152], [278], [306], [59], [165], [48], [203], [217] were also inspired by the milieu of the latter period. The CsH [**105**], [**106**] papers deserve to be singled out in this regard. For instance, the therein presented *U-statistics based processes for detecting a change in the distribution of chronologically ordered observations* have become seminal for further developments along these lines (cf., e.g., Chapter 2 of [A5] and its references, and Markus Orasch [204] in [V2]).

For recent trends and advances on change-point analysis we refer to the four papers of **Part 5** in [V2], and to [**175**]. As an alternative to the sequential empirical process approach (cf. [281]), paper [**175**] with Szyszkowicz introduces the notion of *sequential quantile process* for use in detecting a change in distribution of chronologically ordered independent observations. *Weighted quantile change-point processes* are then studied in this context via approximations in weighted metrics along the lines of [**165**].

Quantile processes, stationary mixing and associated sequences, total time on test processes, empirical Lorenz and Goldie curves. The topics of the monographs [A1], [A2], [A3] continued to be inspiring and were frequently revisited in the years after. We are to view some of these works now.

Initiated by Hao Yu's 1993 thesis [306], *quantile processes based on stationary sequences of random variables* are studied in [**159**] with H. Yu via similarly based Bahadur-Kiefer processes (cf. (2.22)) in properly weighted sup-norm metrics. Thus a general comprehensive approach is provided for obtaining asymptotic results in weighted sup-norm metrics for *uniform quantile processes of stationary sequences*. Consequently, making use of results of H. Yu [307], and Q.–M. Shao and H. Yu [257], the authors establish *weak convergence for weighted uniform quantile processes of stationary mixing and associated sequences*. Next, studying the sup-norm deviation of a general quantile process ρ_n (cf. (2.44)) from its corresponding uniform quantile process (cf. (2.42)) *à la* (2.53)–(2.56) in this context, it is concluded that this weighted sup-norm distance converges in probability to zero under the so-called Csörgő–Révész conditions as in (i), (ii), (iii) (2.47) and (iv) or (v) of (2.54). This, in turn, enables the authors of [**159**] to obtain *weak convergence in weighted sup-norm metrics for general quantile processes* (cf. (2.44)) *of stationary mixing and associated sequences*. In a similar vein, extending corresponding results of [A3] in the i.i.d. case, paper [**163**] with H. Yu establishes a nonparametric large sample estimation theory for *total time on test processes* (cf. (3.12), (3.13) for definitions) *for stationary sequences of positively associated observations*, as well as for those of *ρ– and α–mixing sequences* of random variables. In a similar fashion, extending corresponding i.i.d. based results of [A3], paper [**174**] with Hao Yu constructs a weak approximation theory for *empirical Lorenz curves and their* Goldie [132] *inverses based on a stationary sequence of random variables*. These *approximations* are *also* studied *in terms of stationary sequences of α–mixing random variables*.

The *Lorenz curve* corresponding to a non-negative random variable X with a finite and positive mean $\mu = EX$ and distribution function F, denoted by L_F, is defined (cf. Gastwirth [128]) by the formula

$$L_F(t) := \frac{1}{\mu}\int_0^t F^{-1}(s)ds, \quad 0 \le t \le 1, \tag{3.6}$$

where F^{-1} denotes the left continuous inverse of F (cf. (2.31)). In econometrics it is customary to interpret $L_F(t)$ as the proportion of total amount of "wealth" that is owned by the least fortunate $t \times 100$ percent of a population. In various contexts Lorenz curves have now been in use for just about 100 years (cf. M.C. Lorenz [190], and paper [**157**] with Ričardas Zitikis for further references). Paper [**160**] with Zitikis establishes *asymptotic confidence bands for Lorenz and Goldie curves*, whereas [**169**] with Gastwirth and Zitikis studies such *bands for the Lorenz and Bonferroni curves*. In their survey of convex rearrangements of random elements, Youri Davydov and Ričardas Zitikis [86] in [V2] discuss *relationships of convexifications with generalized Lorenz curves* as well. In particular, related results of [A2], [A3], [**87**], [**157**], [**169**], [**174**] are reviewed in the latter paper [86] in the context of convex rearrangements of random elements.

Mean and percentile residual lifetime processes. G.L. Yang [304], and W.J. Hall and J.A. Wellner in [B1] initiated investigations of the asymptotic uniform behaviour of *mean residual life* (MRL) *processes*. They obtained results holding true over fixed and expanding compact subintervals of $[0, \infty)$ (cf. also [A3]). In [**158**] with Zitikis MRL *processes* are studied *over the whole positive half-line* $[0, \infty)$. Classes of weight functions are introduced that enable the authors to establish (a) *strong uniform*–over–$[0, \infty)$ *consistency* and (b) *weak uniform*–over–$[0, \infty)$ *approximation* of MRL *processes*. The latter in turn leads to constructions of *asymptotic confidence bands for an unknown* MRL *function* M_F, which at (age) $x \geq 0$ is defined by

$$M_F(x) := E(X - x | X > x), \quad x \geq 0, \tag{3.7}$$

where the unknown life distribution function of the nonnegative random variable X is assumed to be continuous with a finite mean EX. The width of the obtained confidence bands is regulated by weight functions depending on information that may be available on the underlying distribution function.

A parallel notion to the MRL function is the *median residual lifetime function* $R^{(1/2)}(t)$, $t \geq 0$. Schmittlein and Morrison [249] explain potential advantages of using the latter instead of MRL. More generally, let F be a life distribution function with the corresponding quantile function $Q = F^{-1}$, defined as in (2.31). With $0 < p < 1$ fixed, consider the $(1-p)$*–percentile residual lifetime* (PRL) *function* at $t > 0$

$$R^{(p)}(t) = Q(1 - p(1 - F(t))) - t, \tag{3.8}$$

originally introduced by Haines and Singpurwalla [141], and interpreted as the $(1-p)$–percentile additional time to failure, given no failure by time $t > 0$. Paper [**91**] with Sándor Csörgő initiates a *nonparametric large sample theory for* PRL *processes*. They introduce the sample analogue $R_n^{(p)}(t)$ of $R^{(p)}(t)$, the *sample* $(1-p)$*–percentile residual life* at $t > 0$, as

$$R_n^{(p)}(t) = Q_n(1 - p(1 - F_n(t))) - t, \quad 0 < p < 1, \tag{3.9}$$

where Q_n and F_n are the sample quantile and empirical distribution functions as in (2.34) and (2.28) respectively, and define the *empirical* $(1-p)$*–percentile life process* $r_n^{(p)}(t)$, *à la* (2.44), by

$$r_n^{(p)}(t) = n^{1/2} f(Q(1 - p(1 - F(t))))(R_n^{(p)}(t) - R^{(p)}(t)), \tag{3.10}$$

where it is assumed that F has density function $f = F'$ that is positive over the support $(0, b)$ of F with b as in (i) (2.47). Paper [**91**] *presents an asymptotic distribution theory for* $r_n^{(p)}(t)$ under conditions that are adapted versions of those of CsR [**40**] (cf. (i)–(iii) (2.47) and (iv), (v) (2.54)) for the general quantile process, resulting in asymptotically distribution-free *confidence intervals* and *confidence bands for the percentile residual lifetime* $R^{(p)}(t)$ of (3.8). The latter parallels those of (2.45) for the quantile function Q as in (2.31). Paper [**82**] concludes *asymptotically* $(1-\alpha)$ *size bands for* $R^{(p)}(t)$ based on *bootstrapped estimates* of $R_n^{(p)}(t)$ (cf. (3.9)). PRL *processes* are studied *under random censorship* in [55]. Further to survival analysis, we refer to Lídia Rejtő and G. Tusnády [222], and H. Yu [309], both in [V1].

Revisiting Lorenz curves and total time on test processes via the general Vervaat process. Paper [**162**] with Zitikis re-examines the topic of [**157**]. Assumi ng the finiteness of the second moment only, it is shown in [**162**] that LIL *for Lorenz curves* holds true provided that the underlying distribution function F and its inverse Q are continuous. Paper [**157**] achieves the same goal with $E|X|^{2+\epsilon} < \infty$ for some $\epsilon > 0$, under the same conditions for F and Q. Earlier LIL results for Lorenz curves also assumed the absolute continuity of F and further assumptions on $f := F'$ as well, together with more than two finite moments (cf. Rao and Zhao [221] and its predecessor [A3, Corollary 11.4, p. 96]). In [**157**] and [**162**] conditions of Rao and Zhao [221] were relaxed via *introducing*, as well as noticing and exploiting the crucial role of, *the general Vervaat process*:

$$V_n^F(t) := \int_0^t (F_n^{-1}(s) - F^{-1}(s))ds + \int_0^{F^{-1}(t)} (F_n(x) - F(x))dx, \quad 0 \le t \le 1, \tag{3.11}$$

where F_n, F_n^{-1} and F^{-1} are as in (2.28), (2.34) and (2.31), respectively. Revisiting the investigations of Section 7.4 of [A2] and Section 6 of [A3] on the uniform rate of convergence to zero of the *total time of test* (TTT) *process* $H_n^{-1} - H_F^{-1}$, where

$$H_F^{-1}(t) := \int_0^{F^{-1}(t)} (1 - F(x))dx, \quad 0 \le t \le 1, \tag{3.12}$$

is the (theoretical) TTT–curve, and

$$H_n^{-1}(t) := \int_0^{F_n^{-1}(t)} (1 - F_n(x))dx, \quad 0 \le t \le 1, \tag{3.13}$$

is its empirical counterpart, the general Vervaat process V_n^F played an equally crucial role in studying the TTT process in [**168**] with Zitikis. For more historical and mathematical details on the Vervaat process we refer to Zitikis [311], and to Wim Vervaat [291], [292], who introduced and studied V_n^F for $[0,1]$–uniform distribution (cf. (3.20) and (3.21)).

Bahadur-Kiefer, Vervaat and Vervaat-error processes in the i.i.d. and long range dependent cases. The fundamental role that was played by the general Vervaat process V_n^F in the papers [**157**], [**162**] and [**168**] has in turn led to studying *Vervaat* and *Vervaat–error processes* on their own in [**172**], [**184**], [**185**], [**188**]. We are to have a glimpse at these papers now. For further details we refer to Csáki et al. [69] in [V2].

With R_n, the (uniform) *Bahadur–Kiefer process* as in (2.22), we have (cf. Kiefer [174], Shorack [260], and Deheuvels and Mason [96])

$$\lim_{n\to\infty} n^{1/4}(\log n)^{-1/2}\frac{\|R_n\|}{(\|\alpha_n\|)^{1/2}} = 1 \quad \text{a.s.}, \tag{3.14}$$

where $\|f\| := \sup_{0\le t\le 1}|f(t)|$ denotes the uniform sup-norm of f and α_n is the uniform empirical process as in (2.13). Now (3.14) in combination with the weak convergence of α_n to a Brownian bridge B yields

$$n^{1/4}(\log n)^{-1/2}\|F_n\| \xrightarrow{\mathcal{D}} (\|B\|)^{1/2}, \quad n \to \infty, \tag{3.15}$$

a result that was first concluded by Kiefer [174] from a convergence in probability version of (3.14).

The asymptotic behaviour of R_n in L_p norms was in turn established in [**195**] with Zhan Shi (cf. also [**183**]) as follows: *For any* $p \in [2, \infty)$, *we have*

$$\lim_{n\to\infty} n^{1/4} \frac{||R_n||_p}{(||\alpha_n||_{p/2})^{1/2}} = c_0(p) \quad a.s., \tag{3.16}$$

where

$$c_0(p) := (E|\mathcal{N}|^p)^{1/p} = \sqrt{2}\left(\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\right)^{1/p} \tag{3.17}$$

with Γ *standing for the Gamma function,* $\mathcal{N}$ *for a standard normal random variable, and* $||f||_p := \Big(\int_0^1 |f(t)|^p dt\Big)^{1/p}$*, the* L_p *norm of* f.

Now (3.16) and the weak convergence of α_n to a Brownian bridge B yield that, for $p \in [2, \infty)$,

$$N^{1/4}||R_n||_p \xrightarrow{\mathcal{D}} c_0(p)(||B||_{p/2})^{1/2}, \quad n \to \infty. \tag{3.18}$$

On the other hand, Vervaat [291], [292] concluded: *The statement*

$$a_n R_n \xrightarrow{\mathcal{D}} Y, \quad n \to \infty, \tag{3.19}$$

cannot hold true in the space $D[0,1]$ (*endowed with the Skorohod topology*) *for any sequence* $\{a_n\}$ *of positive real numbers and any non-degenerate element* Y *of* $D[0,1]$.

Vervaat's [291], [292] proof of this conclusion is based, in a most crucial and elegant way, on the following *integrated Bahadur–Kiefer process*

$$I_n(t) := \int_0^t R_n(s)ds, \quad 0 \le t \le 1. \tag{3.20}$$

Concerning the latter, he established the weak convergence of

$$V_n(t) := 2n^{1/2} I_n(t) \tag{3.21}$$

to B^2, the square of a Brownian bridge (for a discussion and related details we refer to [**172**], **184**]) by showing that

$$\lim_{n\to\infty} ||V_n - \alpha_n^2|| = 0 \quad \text{in probability.} \tag{3.22}$$

Therefore, in the space $C[0,1]$ (endowed with the uniform topology)

$$V_n(\cdot) \xrightarrow{\mathcal{D}} B^2(\cdot), \quad n \to \infty. \tag{3.23}$$

As a consequence of (3.23), Vervaat [291], [292] concluded the impossibility of having (3.19) on account of a Brownian bridge B being almost surely nondifferentiable (cf., e.g., (2.75)).

In retrospect, we note that the impossibility of having (3.19) can also be concluded from a combination of (3.15) and (3.18), or historically more appropriately via (3.15) combined with (cf. Kiefer [172])

$$n^{1/4}R_n(t) \xrightarrow{\mathcal{D}} (t(1-t))^{1/4}\mathcal{N}(|\tilde{\mathcal{N}}|)^{1/2}, \tag{3.24}$$

for every fixed $t \in (0, t)$, where $\mathcal{N}$ and $\tilde{\mathcal{N}}$ are independent standard normal random variables (cf. (2.81), (2.82) for the same phenomenon, and Sections 2 and 3 of Csáki et al. [69] in [V2] for insights into this phenomenon).

As for terminologies, the process V_n of (3.21) is called the *uniform Vervaat process*. The latter coincides with the *integrated empirical difference process* in Shorack and Wellner [263, p. 594]. For further references and elaboration on these terminologies we refer to Section 1 of Zitikis [311].

Bahadur [14] introduced R_n as the remainder term in the representation

$$u_n = -\alpha_n + R_n \tag{3.25}$$

quantile process u_n in terms of the empirical process α_n. In fact, originally, Bahadur [14] studied the latter representation for $\rho_n(y)$ as in (2.44) as a stochastic process in $n \in \mathbb{N}$ for $y \in (0, 1)$ fixed. For a summary along these lines, and for an extension of Kiefer's asymptotic theory of the uniform Bahadur–Kiefer process, we refer again to Chapter 6 of [A2] and that of [A4], papers [**40**], [**64**], [**167**], and [**188**], together with our comments on this matter in the second paragraph that follows right after (2.71) in our exposé.

Back to (3.25), from (3.14), (3.15), (3.24), and from (3.16), (3.18), we see that the remainder term R_n, i.e., the Bahadur–Kiefer process, is asymptotically smaller than the main term α_n, i.e., the empirical process, in both the sup-norm and L_p topologies.

In a similar vein, one can think of the process

$$\hat{V}_n(t) := V_n(t) - \alpha_n^2(t), \quad 0 \le t \le 1, \tag{3.26}$$

that appears in (3.22) as the remainder term $\hat{V}_n$ in the following representation

$$V_n = \alpha_n^2 + \hat{V}_n \tag{3.27}$$

of the uniform Vervaat process V_n in terms of the square of the empirical process. It is well known (cf. Zitikis [311] in [V1] for details and references) that the remainder term $\hat{V}_n$ in (3.27) is asymptotically smaller than the main term α_n^2. Thus, just like in the case of R_n (cf., e.g., (3.14), (3.16), (3.24)), one may like to know how small the remainder term $\hat{V}_n$ is. The latter remainder term $\hat{V}_n$ is called the *Vervaat error process* in [**185**], as well as in [**172**], following the terminology of a 1999 preliminary version of [**185**]. In view of (3.14), (3.16), (3.24), one suspects that there should be substantial differences between the asymptotic pointwise, sup- and L_p–norms behaviour of the process $\hat{V}_n$. Indeed, inspired by [**195**] with Zhan Shi, in [**184**] with Zitikis the following strong convergence result is established for $||\hat{V}_n||_p$: *For any* $p \in [1, \infty)$

$$\lim_{n\to\infty} n^{1/4} \frac{\|\hat{V}_n\|_p}{(\|\alpha_n\|_{3p/2})^{3/2}} = \frac{1}{\sqrt{3}} c_0(p) \quad a.s. \tag{3.28}$$

where $c_0(p)$ *is defined in* (3.17).

For a comparison of this result to that of (3.16), as well as for that of their consequences, we refer to [**184**], where it is also conjectured that in sup-norm the analogue statement of (3.28) should be of the following form:

$$\lim_{n\to\infty} b_n n^{1/4} \frac{\|\hat{V}_n\|}{\|\alpha_n\|^{3/2}} = c \quad \text{a.s.,} \tag{3.29}$$

where b_n is a slowly varying function converging to zero and c is a positive constant. The latter conjecture was proved to be true with $b_n = (\log n)^{-1/2}$ and $c = (4/3)^{1/2}$ in Csáki et al. [**185**], together with asymptotic pointwise results for $\hat{V}_n$ as follows: *For every fixed* $t \in (0,1)$

$$n^{1/4}\hat{V}_n(t) \xrightarrow{\mathcal{D}} (4/3)^{1/2}(t(1-t))^{3/4}\mathcal{N}(|\tilde{\mathcal{N}}|)^{3/2}, \quad n\to\infty, \tag{3.30}$$

$$\limsup_{n\to\infty} \frac{n^{1/4}|\hat{V}_n(t)|}{(\log\log n)^{5/4}} = (t(1-t))^{3/4}\,\frac{2^{11/4}3^{1/4}}{5^{5/4}} \quad \textit{a.s.,} \tag{3.31}$$

where $\mathcal{N}$ *and* $\tilde{\mathcal{N}}$ *are independent standard normal random variables. Also,*

$$\lim_{n\to\infty} n^{1/4}(\log n)^{-1/2}\frac{\|\hat{V}_n\|}{(\|\alpha_n\|)^{3/2}} = (4/3)^{1/2} \quad \text{a.s.} \tag{3.32}$$

As a consequence of this theorem, as well as that of (3.28) combined with (3.32), one arrives at confirming Conjecture 2.1 of [**184**] as follows: *The statement*

$$a_n\hat{V}_n \xrightarrow{\mathcal{D}} Y, \quad n\to\infty, \tag{3.33}$$

cannot hold true in the space $D[0,1]$ (*endowed with the Skorohod topology*) *for any sequence of positive real numbers and for any nondegene rate random element* Y *of the space* $D[0,1]$.

We mention also the following immediate direct consequence of (3.32):

$$n^{1/4}(\log n)^{-1/2}\|V_n\| \xrightarrow{\mathcal{D}} (4/3)^{1/2}||B||^{3/2}, \quad n\to\infty, \tag{3.34}$$

where B *is a standard Brownian bridge.*

For further consequences of (3.32) and their discussion, we refer to [**185**] and Csáki et al. [69] in [V2]. It is of interest to view (3.14) and (3.32) together. One would hope to see one day an L_p–norm version of (3.32) for $\hat{V}_n$ that would rhyme with (3.16) for R_n. The statement of (3.30) is to be compared to (3.24), while that of (3.31) to that of another result of Kiefer [172] for R_n, which reads as follows: *For every fixed* $t \in (0,1)$

$$\limsup_{n\to\infty} \frac{n^{1/4}|\hat{R}_n(t)|}{(\log\log n)^{3/4}} = (t(1-t))^{1/4}\frac{2^{5/4}}{3^{3/4}} \quad \textit{a.s.} \tag{3.35}$$

As noted earlier, the *general Vervaat process* V_n^F (cf. (3.11)) first appeared and was put to good use in [**157**]. For a detailed survey on this subject we refer to Zitikis [311]. For related though rather different limit theorems for the general Vervaat process V_n^F, we refer to [**188**] with Zitikis. It is desirable to generalize the results of (3.30)–(3.32) in such a way that they would cover general forms of the Vervaat process as well. For a recent review of Vervaat and the Vervaat–error processes we refer to [**172**] with Zitikis. The Csáki et al. *paper* [**185**] *is dedicated to the memory of Arthur Hing-Chiu Chan* (1946–1999), Ph.D. 1977, Carleton University.

In [**199**] with Csáki, Rychlik and Steinebach the authors study the asymptotic behaviour of *stochastic processes* that are *generated by sums of partial sums of i.i.d. random variables and their renewals* along the lines of [**185**] (cf. (3.30)–(3.32)). It is concluded that, just like the Bahadur-Kiefer process R_n (cf. (2.22)), the sums of these partial sums and their renewals processes cannot converge weakly to any nondegenerate random element of the space $D[0,1]$. On the other hand, their properly normalized integrals as Vervaat–type stochastic processes are shown to converge weakly to a squared Wiener process. The asymptotic behaviour of the deviations of these processes, i.e., that of their Vervaat–error type processes, is also dealt with in [**199**].

Paper [**198**] with Barbara Szyszkowicz and Lihong Wang establishes strong invariance principles for *Bahadur–Kiefer processes of long range dependent sequences* as in H. Dehling and M.S. Taqu [98]. Moreover, the strong and weak asymptotic behaviour of the integrated Vervaat versions of these processes, as well as that of their deviations from their limits, i.e., that of their Vervaat–error type processes, is also studied in [**198**]. As we have just seen, the Bahadur–Kiefer and the Vervaat error processes cannot converge weakly in the i.i.d. case (cf. (3.19) and (3.33), respectively). In contrast to this, in [**198**] the authors conclude that *the Bahadur–Kiefer and Vervaat–error processes for long range dependent sequences as in* [98] *do converge weakly to a Dehling–Taqqu type limit process* (cf. H. Dehling and M.S. Taqqu [98]).

Spacings, percentile-percentile and quantile rank processes, Chernoff-Savage-type theorems. Statistics based on *spacings* have received a great deal of attention in the literature (cf., e.g., Pyke [218], [219], [**49**], Holst and J.S. Rao [147], Chapter 7 of [A2], Aly [9], Beirlant [18], Aly, Beirlant and Horváth [10], [**79**] with Horváth, and references in these works). Paper [**79**] gives a complete characterization of the asymptotic distribution of the supremum of *weighted spacings*.

Let $X_1,\dots,X_m$ $(m\ge 1)$ and $Y_1,\dots,Y_n$ $(n\ge 1)$ be two independent random samples on the random variables X and Y with respective continuous distribution functions F and G, and let F_n and G_n denote the corresponding empirical distribution functions. Put $N=m+n$, $\lambda_N=m/N$, $H=\lambda_N F+(1-\lambda_N)G$, $H_N=\lambda_N F_m+(1-\lambda_N)G_n$. Let F^{-1}, F_m^{-1}, etc., denote the inverse functions of F, F_m, etc. Paper [**86**] with Aly and Horváth studies the following processes: *the empirical* P–P (*procentile–procentile*) *process*

$$\ell_N(y)=N^{1/2}(G_nF_m^{-1}(y)-GF^{-1}(y)),\quad 0\le y\le 1, \tag{3.36}$$

the *quantile rank process*

$$d_N(y)=N^{1/2}(D_N(y)-D(y)),\quad 0\le y\le 1, \tag{3.37}$$

where $D_N(y)=H_NF_m^{-1}(y)$, $D(y)=HF^{-1}(y)$, and the *empirical rank process*

$$r_n(y)=N^{1/2}(D_N^{-1}(y)-D^{-1}(y)),\quad 0\le y\le 1. \tag{3.38}$$

Weak convergence of these processes to appropriate Gaussian processes is established under certain regularity conditions. Let R_k stand for the rank of the kth order statistic in the pooled sample. Then $nG_n(X_{k,n})=R_k-k$, $k=1,\dots,m$. This relationship shows that ranks provide an alternative way of looking at P–P plots. It was the latter connection and Parzen's [210] inspiring expository paper on two sample statistics that have led us to also study rank plots and rank processes in [**86**]. Pyke and Shorack [220] considered the process $N^{1/2}(F_mH_N^{-1}(y)-$

$FH^{-1}(y))$, $0 \le y \le 1$, which is asymptotically equivalent to r_n of (3.38). They proved its weak convergence in weighted metrics to a Gaussian process. The approach of [**86**] to rank processes *à la* r_n of (3.38) is more direct and lends itself to bootstrapping immediately via Section 17 of [A3]. The main tools are weighted approximations of the uniform empirical and quantile processes *à la* Pyke and Shorack [220] and [**74**]. The obtained results in weighted metrics are directly used to prove *Chernoff-Savage-type theorems* (H. Chernoff and I.R. Savage [51]).

Bootstrapped empiricals, estimating the tail index of distribution and the adjustment coefficient in risk theory via intermediate order statistics. Having just mentioned bootstrapping r_n of (3.38) via Section 17 of [A3], we note that the $O(n^{-1/4}(\log n)^{3/4}$ rate of *approximation of the bootstrapped empirical process* as in [A3] is improved in [**177**] with Horváth and Kokoszka, where a KMT [181] type approximation (cf. (2.15)) is obtained with the same rate of coupling for the bootstrapped empirical process. This construction in turn was shown to be best possible by Horváth and Steinebach [149]. The proof of the new approximation is based on the Poisson approximation of the uniform empirical distribution function by Bretagnolle and Massart [36] (cf. also Theorem 3.1.3 in [A4, p. 139]) and the Gaussian approximation for randomly stopped sums of paper [**90**] with Deheuvels and Horváth (cf. also [**81**] and [**92**] with Horváth and Steinebach, and Chapter 2 of [A4]).

Somewhat in isolation, paper [**89**] with Horváth and Révész deals with the elusive problem of *estimating the tail index of a distribution.* A straightforward (naive) estimator is proposed, whose properties are examined for consistency and asymptotic normality. Its various rates of convergence are explored under different conditions. It is seen to compare well with other tail index estimators found in the literature. Using the naive estimator, optimality is explored. It is concluded that, while optimal rates of convergence do exist under various conditions, the notion of an optimal sequence for this problem is bound to run into unsurmountable difficulties. The same is shown to be true for the more complex, previously introduced estimators. For a comprehensive review of this area, we refer to Sándor Csörgő and László Viharos [81] with 138 reference works in [V1]. In connection with [**89**] they note that, without investigating it, Gawronski and Stadtmüller [129] also proposed the same naive estimator, while A.H. Welsh [299] studied essentially the same estimator in the context of P. Hall's paper [143].

In a somewhat similar vein to that of [**89**], in [**121**] with Josef Steinebach a sequence of *intermediate order statistics* is proposed to *estimate* the *adjustment coefficient in risk theory.* The underlying random variables may be viewed as *maximum waiting times in busy cycles of* GI/G/1 *queuing models under light traffic.* The estimates are shown to be strongly consistent and their rates of convergence is also studied. There are also *central limit theorems* available *for intermediate order statistics* (cf. Leadbetter et al. [185, Chapter 2.7] and Cooil [58]). The latter were extended in terms of weak convergence in weighted metrics in CsH [**84**].

Randomly stopped sums, Rayleigh random flight, stochastic analysis, geometric stochastic processes, Black-Scholes equations. The *strong approximation* theory *of randomly stopped sums* of i.i.d.r.v.'s as in [**90**] with Deheuvels and Horváth is applied in the same paper to studying *Kingman's* GI/G/1 *queues in heavy traffic* (cf. J.F.C. Kingman [176], [177]). For earlier results on the accuracy of Kingman's approximation we refer to Rosenkrantz [244] and Kennedy [164]. The problem of providing *estimates for the probability of ruin* starting with a large initial reserve as in Horváth and Willekens [150] is also revisited in [**90**], improving on results of the latter paper. For more details and further references on the just mentioned topics we refer to Sections 2.4.1–2.4.3 of [A4]. In connection with *renewal counting and first passage* *time processes* we call attent ion to Allan Gut and Josef Steinebach [138] in [V2], studying these topics in terms of *complete convergence*, following their survey of similar asymptotics for partial sums. Sándor Csörgő [78] in [V2] determines the *complete convergence of bootstrap means.* We note in passing that the notion of *complete convergence* has also played a significant role in proving *random limit theorems via strong invariance principles* (cf. Chapter 7 of CsR [A1]).

Further to randomly stopped sums, let $\xi, \xi_1, \xi_2, \dots$ be i.i.d.r.v.'s with $E\xi > 0$ and $0 < \mathrm{Var}\xi < \infty$. Define

$$A(t) = \inf\{k : \sum_{i=1}^{k} \xi_i > t\} \text{ and } Y(t) = \sum_{i=1}^{A(t)-1} (-1)^{i+1}\xi_i, \quad t \geq 0. \tag{3.39}$$

Assuming that ξ is an exponential r.v., $Y(t)$ is called the *transport process* or *Rayleigh random flight*. With the help of [**92**], CsH [**104**] obtain a rate of approximation of $Y(t)$ by a Wiener process. The latter rate of approximation is the same as the optimal rate of approximation in [**90**], where, unlike in [**104**], the summands and the stopping processes are assumed to be independent. This approximation is applied to the E. Wong and M. Zakai [302], [303] *approximation of stochastic integrals*. For further details and references on these approximations we refer to Luis G. Gorostiza's review of [**104**] (cf. **MR 89k:60073**) and to Section 2.4.4 of [A4], where ξ above is assumed to have a finite moment generating function in a neighbourhood of zero, instead of being exponentially distributed as in [**104**].

The survey paper [**173**] *presents* the elements of mathematics that are relevant to financial modeling in a historical context. Using results of Tamás Szabados [275], [276], it gives a *self-contained background on stochastic analysis* (Itô calculus included), and also *deals with the problem of fair pricing of financial derivatives and their related Black-Scholes formulas.* The results of the paper [**176**] are also previewed in this context. For a glimpse of the latter paper we refer to Csáki et al. [69] in [V2].

Weak and strong approximations for logarithmic averages. The discovery of the pointwise central limit theorem (cf. G.A. Brosamler [38], P. Schatte [248]) created a considerable interest in logarithmic limit theorems. CsH [**132**] obtain *weak and strong Gaussian approximations for logarithmic averages* of indicators of normalized partial sums of i.i.d. random variables with $EX = 0$, $EX^2 = 1$, $EX^2(\log|X|)^{2+\delta} < \infty$ for some $\delta > 0$. The proofs are based on *invariance principles for integrals of an Ornstein–Uhlenbeck process and on strong approximations of normalized partial sums* by *Ornstein–Uhlenbeck processes.* A. Weigl [297] proved the first CLT and LIL in this context for a symmetric random walk. István Berkes and Lajos Horváth [21] extend the results of CsH [**132**] to a larger class of independent sequences. For an insightful comprehensive review of results and problems related to pointwise central limit theorems we refer to Berkes [20] in [V1].

Pre-super Brownian motion. Super Brownian motion can be defined as a high-density limit of critically branching Brownian motions. The paper CsR [**180**] uses the term *pre-super Brownian motion* to refer to *the branching particle system before the limiting process.* First, a sequence of results is proved on binary branching Brownian motion, where the branching is controlled by a sequence of i.i.d. uniform–$(0,1)$ random variables. Namely, a new particle born at time t is replaced by 2 independent particles at time $t + (1-t)U$, where $U \stackrel{\mathcal{D}}{=\!=}$ uniform $-(0,1)$. John Verzani notes in his review **MR 2002f:60163** that this process, sometimes called a "stick-braking process", has been used in connection with the superprocess previously (cf. A.M. Etheridge [118, Section 3.4], and references therein). In [**180**], the thus obtained results are related to the pre-super Brownian motion and its limit.

Local times in a random environment. We are now to have a look at *local times in a random environment*. As to the *random environment*, let $\mathcal{E} = \{E_i,\ i \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables with distribution function $F(x) = P(E_0 \le x)$, $0 < x < 1$, $F(0) = 0$, $F(1) = 1$. Any realization of this random environment $\mathcal{E}$ will be denoted by the same letter. *For any fixed sample sequence of this environment*, define a random walk $\{S_n\}$ by $S_0 = 0$ and $P_{\mathcal{E}}\{S_{n+1} = i+1 | S_n = i\} = 1 - P_{\mathcal{E}}\{S_{n+1} = i-1 | S_n = i\} = E_i$, $n = 0, 1, \ldots,\ i = 0, \pm 1, \ldots$. This sequence $\{S_n\}$ is called a *random walk in random environment* (RWIRE) and, initiated by F. Solomon [268], it has been widely studied under the following conditions with respect to the distribution P of E_0:

$$\text{There is an} \quad 0 < a < 1/2 \quad \text{such that} \quad P\{a < E_0 < 1 - a\} = 1, \tag{3.40}$$

$$E \log \frac{E_0}{1 - E_0} = 0, \tag{3.41}$$

$$0 < \sigma^2 := E \log^2 \frac{E_0}{1 - E_0} < \infty. \tag{3.42}$$

These *conditions guarantee the recurrence of* $\{S_n\}$.

F. Solomon [268] proved that under these conditions

$$P\{P_{\mathcal{E}}(S_n = 0 \text{ i.o. }) = 1\} = 1, \tag{3.43}$$

i.e., *for almost all realizations of the random environment, the particle returns to the origin infinitely often with probability one.*

The celebrated limit theorem of Ya. G. Sinaĭ [265] in the recurrent case under the conditions (3.41), (3.42) concludes that *for each realization of the random environment* $\mathcal{E}$ *there exists a function* s_n *such that, as* $n \to \infty$, $(\log n)^{-2}(S_n - s_n)$ *converges in probability to* 0, *and that* $(\log n)^{-2}\sigma^2 s_n$ *converges in distribution to a random variable L which is a functional of a Wiener process.* H. Kesten [168] succeeds in computing the distribution of L exactly via showing that *L has an explicitly given density.*

Thus, unlike in the classical case, for large n, a random walk in random environment behaves like $(\log n)^2$. In fact, the same holds true for $M(n) := \max_{0 \le k \le n} |S_k|$ as well. Indeed, it is shown in P. Deheuvels and P. Révész [97] that, on assuming the conditions (3.41), (3.42), one has *for any* $\epsilon > 0$

$$(\log n)^2 (\log\log n)^{-2-\epsilon} \le M(n) \le (\log n)^2 (\log\log n)^{2+\epsilon}, \tag{3.44}$$

provided that $n \ge n_0$, *where* $P_{\mathcal{E}}\{n_0 < \infty\} = 1$ a.s.

Let the *local time of a* RWIRE $\{S_k\}$ up to n be $\xi(x, n) = \#\{k : 0 \le k \le n, S_k = x\}$. Since *in the classical case* $M(n)$ is "around" $n^{1/2}$, it is not surprising that, asymptotically in n, $\xi(x, n)$ as in (2.78) is also like $n^{1/2}$ (cf. K.L. Chung and G.A. Hunt [57], E. Csáki and P. Révész [73], and G. Simons [264]). *In the random environment case the local time fluctuation is much bigger.* For example, on assuming the conditions (3.40)–(3.42), one has (cf. P. Deheuvels and P Révész [97], and P. Révész [240]) for any fixed integer k and $\varepsilon > 0$,

$$P_{\mathcal{E}}\left\{\xi(k, n) \le \exp\left(\frac{\log n}{(\log\log n)^{1-\epsilon}}\right) \text{ i.o. }\right\} = 1 \quad \text{a.s.}, \tag{3.45}$$

and

$$\xi(k, n) \ge \exp\left(\frac{\log n}{(\log\log n)^{1+\epsilon}}\right), \tag{3.46}$$

provided $n \geq n_0$, where $P_{\mathcal{E}}\{n_0 < \infty\} = 1$ a.s.

In the light of this big fluctuation via large n, it is natural to expect that the *local time* $\xi(k,n)$ *in* k *is also much less stable in the case of* RWIRE than in the classical situation. For the sake of describing stability, just like in the classical case, consider the ratio

$$\nu(k,n) := \frac{\xi(k,n)}{\xi(0,n)}. \tag{3.47}$$

In the classical case, i.e., *when* $\xi(\cdot,\cdot)$ *is as defined as in* (2.78), one has (cf. CsR [**69**], and E. Csáki and A. Földes [68]) *for any* $\epsilon > 0$

$$\lim_{n\to\infty} \sup_{|k|\leq n^{1/2}/(\log n)^{1+\epsilon}} |\nu(k,n) - 1| = 0 \quad \text{a.s.} \tag{3.48}$$

and

$$\lim_{n\to\infty} \sup_{|k|\leq n^{1/2}/\log n} |\nu(k,n) - 1| \geq 1 \quad \text{a.s.} \tag{3.49}$$

Intuitively it is clear that *for* RWIRE *one cannot expect that* $\nu(k,n)$ *would converge to* 1 even for a fixed k. In paper [**95**] with P. Révész and L. Horváth it is shown however that there exists a random function $m(k,\mathcal{E}) = m_k$ which depends only on the environment $\mathcal{E}$, i.e., $m(k,\cdot)$ is a deterministic function for any realization of the environment $\mathcal{E}$, such that $\nu(k,n)$ *is near* m_k *in the recurrent case.* Namely, on assuming the conditions (3.41) and (3.42), one has for *any* $\epsilon > 0$ *and* k *a fixed integer*

$$P_{\mathcal{E}}\left\{\lim_{n\to\infty} \exp\left((\log n)(\log\log n)^{-(1+\epsilon)}\right)|\nu(k,n) - m_k| = 0\right\} = 1 \quad \text{a.s.} \tag{3.50}$$

The result in (3.44) indicates that in the statement of (3.50) one cannot expect uniformity in k when k is very close to $(\log n)^2$. Indeed, as *an analogue of* (3.48) *and* (3.49) *combined*, paper [**95**] also concludes that, *on assuming* (3.41), (3.42) *together with*

$$F(x) + 1 - F(1-x) = O(x^{\alpha}) \text{ with some positive } \alpha \text{ as } x \to \infty, \tag{3.51}$$

then, for each $\epsilon, \delta > 0$,

$$\begin{aligned} P_{\mathcal{E}}\Big\{ &\lim_{n\to\infty} \exp\left((\log n)(\log\log n)^{-(1+\delta)}\right) \\ &\times \max_{1\leq k\leq (\log n)^2/(\log\log n)^{2+\epsilon}} |\nu(k,n) - m_k| = 0\Big\} = 1 \quad a.s., \end{aligned} \tag{3.52}$$

and for each $C > 1/(2\sigma^2)$ *with some* $\epsilon = \epsilon(C) > 0$

$$P_{\mathcal{E}}\Big\{\limsup_{n\to\infty} n^{-\epsilon} \max_{1\leq k\leq C(\log n)^2/\log\log n} |\nu(k,n) - m_k| = \infty\Big\} = 1 \quad \text{a.s.} \tag{3.53}$$

The fluctuation of m_k and, via (3.52) also that of $\nu(k,n)$, is very big. As to the question of how big, paper [**95**] concludes that, *under the conditions* (3.41), (3.42) *and* (3.51), *with a sequence of positive integers* $\{k_n\}$ *such that, as* $n \to \infty$,

$$\frac{(\log\log n)^{2+\epsilon}}{(\log n)^2} k_n \to 0 \text{ for some } \epsilon > 0 \text{ and } k_n \to \infty, \tag{3.54}$$

one has, for any $x > 0$,

$$\lim_{n\to\infty} P\Big\{(\nu(k_n,n))^{k_n^{-1/2}} \le x\Big\} = \Phi\Big(\frac{1}{\sigma}\log x\Big), \tag{3.55}$$

where Φ *is the standard normal distribution function.* Moreover, *if in addition to* (3.41), (3.42), (3.51) *and* (3.54),*one also assumes that* k_{n+1}/k_n *is bounded, then the* LIL *that corresponds to* (3.55) *reads as follows*

$$P_{\mathcal{E}}\Big\{\limsup_{n\to\infty}(\nu(k_n,n))^{(2k_n\log\log k_n)^{-1/2}} = e^{\sigma}\Big\} = 1 \quad \text{a.s.}, \tag{3.56}$$

$$P_{\mathcal{E}}\Big\{\liminf_{n\to\infty}(\nu(k_n,n))^{(2k_n\log\log k_n)^{-1/2}} = e^{-\sigma}\Big\} = 1 \quad \text{a.s.} \tag{3.57}$$

For many more results and much more insight into RWIRE and related matters, we refer to Part III of Révész [242]. Especially for studying the RWIRE local time $\xi(0,n)$ and the favourite value of the RWIRE, we refer to paper [241], and, respectively, to Chapters 28 and 29 of [242]. *In particular*, we wish to call attention to *the beautiful main result of* [241] (cf. also Theorem 29.1 of Chapter 29 of [242]) which, for the maximum local time of a RWIRE, $\xi(n) := \max_k \xi(k,n)$, that is based on the *specific random environment* $\mathcal{E} = \{E_i,\ i \in \mathbb{Z}\}$ *with distribution* $P(E_0 = p) = P(E_0 = 1-p) = 1/2$, $0 < p < 1/2$, *concludes that there exists a constant* $g = g(p) > 0$ *such that* $\limsup_{n\to\infty} n^{-1}\xi(n) \ge g(p)$ a.s. *for almost all realizations of this random environment*, i.e., $P\{P_{\mathcal{E}}(\limsup_{n\to\infty} n^{-1}\xi(n) \ge g(p)) = 1\} = 1$. In his Remark 1 on page 298 of [242] Révész notes that, very *likely*, the latter *conclusion remains true under the usual conditions* (3.40)–(3.42) as well. These conditions are of course *implied by the just mentioned specific random environment assumptions*. So, *in random environment the local time can* indeed *be very big*.

Brownian local time distributions, moduli of continuity for local times of Gaussian processes. *Further to local times*, let $\{W(t);\, t \ge 0\}$ be a standard Wiener process and let $\{L(x,t);\, x \in \mathrm{R}^1,\, t \ge 0\}$ be its local time as in (2.79). Paper [**140**] with Qi-Man Shao provides a new proof for the *distribution of* $L(x,t+h) - L(x,t)$ for each $x \in \mathrm{R}^1$, $t \ge 0$, $h > 0$. Using the Fourier analytic approach to the local time, due to Berman [24], [26], the proof involves the *explicit computation of the* mth *moment* of the latter random variable. In particular, the result itself, for example yields $L(0,t) \stackrel{\mathcal{D}}{=} \sup_{0\le s\le t} W(s)$ (cf. P. Lévy [186]), as well as (cf. P. Lévy [187])

$$P\{L(0,t+h) - L(0,t) = 0\} = \frac{2}{\pi}\tan^{-1}(\sqrt{t/h}). \tag{3.58}$$

Let $\{X(t);\, t \ge 0\}$ be a real-valued stochastic process with occupation time

$$H(A,t) = \lambda\{s : 0 \le s \le t,\ X(s) \in A\}, \quad t \ge 0, \tag{3.59}$$

for any Borel set A of the real line, where λ is the Lebesgue measure. If, for each fixed t, $H(\cdot,t)$ is absolutely continuous with respect to Lebesgue measure, then its Radon-Nikodym derivative is called the local time (occupation density) of $X(\cdot)$ at t, denoted by $\ell(\cdot,t)$. Then, *à la* (2.79), $H(A,t) = \int_A \ell(x,t)dx$. In [**151**] with Zheng-Yan Lin and Qi-Man Shao the authors study the sample path properties of increments in t of the *local times* $\ell(x,t)$ *of Gaussian processes with stationary increments* and those *of stationary Gaussian processes* as well, both for fixed x and for all x. These investigations are guided by some well known fine analytic properties of the moduli in t of the local time $L(x,t)$ of a standard Wiener process (cf. J. Hawkes [145], E. Perkins [213], Csáki et al. [**58**], H. Kesten [167], Csáki et al. [69, Theorem 2.1] in [V2]). The *main condition* employed in the paper was first introduced in local time theory for Gaussian processes by Simeon M. Berman [25]. The condition *is that the incremental variance function* $\sigma^2(\cdot)$ *is continuous and concave on some interval* $[0,T]$, $T > 0$. This condition facilitates the computation of the determinant of the covariance matrix of the finite-dimensional distribution of the process that arises in the expression for the higher moments of $\ell(\cdot,\cdot)$ (cf. Lemma 3.5 of [**151**]). An example of one of the several conclusions reads as follows (cf. Corollary 2.2 to Theorem 2.1 of [**151**]): *Let* $\{X(t),\, t \geq 0\}$ *be a fractional Wiener process of order* α*,* $0 < \alpha \leq 1/2$*, i.e., a centered Gaussian process with stationary increments and* $\sigma^2(h) = E(X(t+h) - X(t))^2 = h^{2\alpha}$*,* $t,h \geq 0$*. Then*

$$\limsup_{h \downarrow 0} \frac{\ell(0,h)}{h^{1-\alpha}(\log\log(1/h))^{\alpha}} \leq 200 \quad \text{a.s.} \tag{3.60}$$

In case of a standard Wiener process $W(\cdot)$ with local time $L(\cdot,\cdot)$, (3.60) with $\alpha = 1/2$ reads

$$\limsup_{h \downarrow 0} \frac{L(0,h)}{(h\log\log(1/h))^{1/2}} \leq 200 \quad \text{a.s.} \tag{3.61}$$

The local version of Kesten's LIL [167] yields

$$\limsup_{h \downarrow 0} \frac{L(0,h)}{(2h\log\log(1/h))^{1/2}} = 1 \quad \text{a.s.}$$

Consequently, the bound one gets in (3.61) from (3.60) via Theorem 2.1 of [**151**] that deals with mean zero Gaussian processes of stationary increments in general, is of the precise order.

Infinite dimensional Ornstein–Uhlenbeck processes, Banach space valued processes: the limit superior behaviour of their path increments. Let

$$\{Y(t),\, t \in \mathrm{R}^1\} = \{X_k(t),\, t \in \mathrm{R}^1,\, k = 1,2,\ldots\} \tag{3.62}$$

be a *sequence of independent Ornstein–Uhlenbeck processes* with coefficients γ_k and λ_k, i.e., $X_k(\cdot)$ is a stationary mean zero Gaussian process with covariance function

$$EX_k(s)S_k(t) = (\gamma_k/\lambda_k)\exp(-\lambda_k|t-s|), \quad (\gamma_k, \lambda_k > 0,\, k = 1,2,). \tag{3.63}$$

The process $Y(\cdot)$ was first studied by Don Dawson [87] as the stationary solution of the infinite array of stochastic differential equations

$$dX_k(t) = -\lambda_k X_k(t)dt + (2\gamma_k)^{1/2} dW_k(t), \quad k = 1,2,\ldots, \tag{3.64}$$

where $\{W_k(t),\, t \in \mathrm{R}^1\}$ are independent Wiener processes (cf. also D.A. Dawson [88], J.B. Walsh [294], A. Antoniadis and R. Carmona [13]). The *continuity properties* of $Y(\cdot)$ were investigated by D.A. Dawson [87], I. Iscoe and D. McDonald [153], [154], B. Schmuland [250], [251], [252], and Iscoe et al. [155]. The final result in this regard is due to X. Fernique [124]. *Moduli of continuity for* $Y(\cdot)$ were first studied by B. Schmuland [251] and in the paper [**103**] with Z.Y. Lin. Further along these lines, path properties of $Y(\cdot)$ were studied in [**102**], [**114**], [**115**] with Zhengyan Lin,

X. Fernique [124], [125], [126], I. Iscoe et al. [155], B. Schmuland [253], [254], Csáki et al. [**125**]. For example with $X_k(\cdot)$ as in (3.62)–(3.63), in [**115**] the authors study the real valued processes

$$X(t) := \sum_{k=1}^{\infty} X_k(t) \quad \text{and} \quad \chi^2(t) := \sum_{k=1}^{\infty} X_k^2(t), \quad t \in \mathrm{R}^1, \tag{3.65}$$

along the lines of CsR [**43**] and Chapter 1 of CsR [A1]. On the other hand, Csáki et al. [**125**] first establish moduli of *continuity and large increment properties for* more general *mean zero stationary increment Gaussian processes* in order to study the path behaviour of $X(\cdot)$ as in (3.65). The existence and continuity of such infinite series type Gaussian processes are proved via showing that under a global condition their corresponding partial sums processes

$$\{X(t,n) := \sum_{k=1}^{n} X_k(t),\ t \in \mathrm{R}^1,\ n = 1,2,\ldots\} \tag{3.66}$$

converge uniformly over finite intervals with probability one. In [**114**] the authors study $Y(\cdot)$ of (3.62) in terms of the two-time parameter stochastic process $X(\cdot,\cdot)$ as in (3.66) and in that of its version when both time parameters are continuous (cf. (3.79)). The methods used and the results obtained in this regard are similar to those used in Chapter 1 of CsR [A1] when studying the Wiener sheet. We also note in passing that in the papers [**115**] and [**125**] there is an exhaustive description of related results available.

The papers [**119**], [**122**], [**130**], [**133**] initiate and extend *investigations on moduli of continuity and large increments to Banach space valued processes, with special attention to* ℓ^2*– and* ℓ^p*–valued processes.* Adapting to style of referencing as in our present presentation, we quote from [**130**] with Endre Csáki:

> The essence of our approach is the realization that the inequalities of Lemmas 1.1.1 and 1.2.1 for increments of a standard Brownian motion in Csörgő and Révész [A1] [cf. also Lemmas 1 and 1* of Csörgő and Révész [**43**]] can be extended to increments of general, nonstationary, not necessarily Gaussian, Banach space valued processes, def ined on the real line.

With reference to [**130**], and [**133**] with Q.-M. Shao, this generality is achieved by first assuming specific conditions on the tail behaviour of the increments of $\{\Gamma(t),\ t \in \mathrm{R}^1\}$, a stochastic process with values in a separable Banach space $\mathcal{B}$ with norm $||\cdot||$ that is assumed to be almost surely continuous with respect to the latter norm. Based on such tail conditions, *large deviation results* are proved *for increments like* $\sup_{0\le t\le T-a}\sup_{0\le s\le a}||\Gamma(t+s)-\Gamma(T)||$ (cf., e.g., Theorem 5.1 in Csáki et al. [69] in [V2]), which then are used to establish *moduli of continuity and large increment estimates* for $\Gamma(\cdot)$ (cf., e.g., Theorem 5.2 in Csáki et al. [69] in [V2]). One of the many applications is to prove moduli of continuity estimates for ℓ^2–valued and ℓ^2–norm squared Ornstein-Uhlenbeck processes, as in [**130**] and [**133**] for example. One of the motivations for studying the ℓ^2–norm squared real valued process $\chi^2(\cdot)$ as in (3.65) is due to the observation that $Y(\cdot)$ of (3.62)–(3.63) in ℓ^2 is almost surely continuous if and only if $\chi^2(\cdot) \in \mathrm{R}^1$ is almost surely continuous (cf., e.g., Lemma 5.1 of [**130**]). Paper [**150**] follows the general pattern of [**130**] and [**133**] in studying ℓ^p–valued Gaussian processes for $1 \le p \le 2$. While for any given process the assumed tail conditions may not always be easy to check, the idea is to separate out the two parts of the standard proofs of stochastic moduli results: (i) the tail behaviour and large deviations of increments, and (ii) their applications to get the moduli results. The *setup is general*, and *there is no need for Gaussian, or Gaussian related, restrictions.* For related results on ℓ^p–valued Ornstein-Uhlenbeck processes, with $p \geq 1$, we refer to Fernique [125], [126], and with $1 \leq p \leq 2$, to Schmuland [254] and [**119**] with Csáki.

The moduli of continuity studies of [**133**] on the real valued ℓ^2–norm squared process $\chi^2(t)$ as in (3.65) are revisited in [**136**], where the authors establish *exact randomized moduli of continuity* for the latter process. For example, it is shown that under suitable conditions on the γ_k and λ_k in (3.63), one has

$$\lim_{h\downarrow 0} \sup_{0\leq t\leq 1} \sup_{0\leq s\leq h} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h)(2\log\frac{1}{h})^{1/2}} = 1 \quad \text{a.s.} \tag{3.67}$$

and, for each t,

$$\limsup_{h\downarrow 0} \sup_{0\leq s\leq h} \frac{|\chi^2(t+s)-\chi^2(t)|}{\Gamma(t,h)(2\log\log\frac{1}{h})^{1/2}} = 1 \quad \text{a.s.,} \tag{3.68}$$

where $\Gamma^2(t,h) := 4\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k}(1-e^{-2\lambda_k h})X_k^2(t)$, t, $h \geq 0$.

It is inviting to compare the randomized moduli of continuity results in (3.67) and (3.68) with P. Lévy's moduli for Brownian motion (cf., e.g., Theorems 1.1.1, 1.3.3 and 1.3.3* in CsR [A1], and Section 5 of Khoshnevisan [171] in [V2] for recent developments along these P. Lévy moduli lines). These results also have the characteristics of the strong increment theorems for self-normalized partial sums (cf. Csörgő et al. [**192**, Section 6] in [V2]).

Paper [**138**] with Qi-Man Shao starts with *combining* Lemma 2.1 *and* Theorem 3.1 of Csáki et al. [**133**] (cf. Theorems 5.1 and 5.2 respectively, in Csáki et al. [69] in [V2]) *as Theorem* A *for use as a general estimate for studying increments of Banach space valued processes.* As summarized just above when viewing papers [**130**], [**133**] and [**150**] together, the novelty and usefulness of this combination of Lemma 2.1 and Theorem 3.1 of [**133**] alone yielded significant new results when studying various path properties of ℓ^p–valued, $1 \leq p \leq 2$, Gaussian processes in Csáki et al. [**133**] and [**150**]. In [**138**] with Qi-Man Shao *Theorem* A *is combined with* the well-known *Borell inequality* as given in Robert J. Adler [1], via the dual-space idea as used in Michael B. Markus and Jay Rosen [194], and thus it succeeds in refining the earlier results of Csáki et al. [**133**], [**150**], as well as in extending them to ℓ^p-valued, $1 \leq p < \infty$, Gaussian processes with stationary increments. The use of this approach is demonstrated in proving LIL results for ℓ^p-valued $1 \leq p < \infty$ Gaussian processes, as well as for studying fine sample path properties of ℓ^p-valued, $1 \leq p < \infty$, *fractional Wiener and fractional Ornstein-Uhlenbeck processes* (cf. Sections 5 and 6 respectively, of [**138**]). Robert J. Adler in his **MR 95k: 60084** writes: "The proofs hinge on delicate inequalities for Gaussian, Banach space valued processes, as well as the usual tools of the Gaussian theory, including Borell's inequality." A particular case of (5.7) of Theorem 5.1, that may be of independent interest, illustrates the powerful and far reaching nature of the results of [**138**]: *Let* $\mathbf{W}(t) = (W_1(t), \ldots, W_d(t))$, $t \geq 0$, *be a standard* d*–dimensional Wiener process. Then*

$$\limsup_{T\to\infty} \frac{\left(\sum_{i=1}^{d} |W_i(t)|^p\right)^{1/p}}{(2T\log\log T)^{1/2}} = d^{(2-p)/2p} \quad \textit{a.s., if} \;\; 1 \leq p < 2, \tag{3.69}$$

and

$$\limsup_{T\to\infty} \frac{\left(\sum_{i=1}^{d} |W_i(t)|^p\right)^{1/p}}{(2T\log\log T)^{1/2}} = 1 \quad \textit{a.s., if} \;\; p \geq 2. \tag{3.70}$$

Csörgő et al. [**145**] establish moduli of continuity results for ℓ^∞ valued continuous Gaussian processes with stationary increments in general, as well as for ℓ^∞ valued Ornstein-Uhlenbeck processes in particular, namely, in both cases, the size of increments of the type $\sup_{k\geq 1}|X_k(t+s)-X_k(t)|$ are analyzed uniformly in $t\in[0,1]$ and $s\in[0,h]$ as $h\downarrow 0$.

For recent developments and further references that are related to our discussion starting with the papers [**119**], [**122**], [**130**], [**133**] above, we refer to Yong-Kab Choi [53].

Limit inferior behaviour of inf-sup increments of Banach space valued stochastic processes. Paper [**144**] with Qi-Man Shao establishes a *criterion for the limit inferior behaviour* of $\inf_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\|\Gamma(t+s)-\Gamma(t)\|$, where $\{\Gamma(t),\, t\in \mathrm{R}^1\}$ is a stochastic process with values in a separable Banach space $\mathcal{B}$ with norm $\|\cdot\|$. The idea of the latter criterion in [**144**] is similar to that of the above discussed Theorem A in [**138**] (cf. Theorems 5.1 and 5.2 combined in Csáki et al. [69] in [V2]), whose usefulness was demonstrated in [**133**], [**150**] and [**138**] for studying path properties of large and small increments of ℓ^p–valued, $1\leq p<\infty$, Gaussian processes. The similar criterion of [**144**] for the limit inferior behaviour of $\inf_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\|\Gamma(t+s)-\Gamma(t)\|$ is related to the notion of non-differentiability of $\Gamma(\cdot)$, and it reads as follows (cf. Theorem 2.1 of [**144**]): *Let a_T, b_T be nonnegative continuous functions, and $v(t)$ be a nonnegative monotone nondecreasing function. Assume that there exist positive constants c and d such that for each $t>0$*

$$\frac{1+b_T}{a_T}+a_T\to\infty \quad as \quad T\to\infty, \tag{3.71}$$

$$P\Big\{\sup_{0\leq s\leq \frac{a_T}{2}}\|\Gamma(t+s)-\Gamma(t)\|\leq v(x)\Big\}\leq c\exp(-da_T/x) \tag{3.72}$$

for each $t\in[0,2b_T]$ and

$$\frac{da_T}{4(\log(b_T/a_T)+\log\log\tilde{a}_T}\leq x\leq\frac{4da_T}{\log(b_T/a_T)+\log\log\tilde{a}_T},$$

where $\tilde{a}_T=a_T+1/a_T$. Then

$$\liminf_{T\to\infty}\inf_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\frac{\|\Gamma(t+s)-\Gamma(t)\|}{v(da_T/2(\log(b_T/a_T)+\log\log\tilde{a}_T))}\geq\frac{1}{2}\quad a.s. \tag{3.73}$$

Clearly, if $a_T\to 0$ or $a_T\to\infty$, or $b_T\to\infty$ as $T\to\infty$, then (3.71) is satisfied. Consequently, (3.71) *includes the usual large and small increments as special cases.*

The following example shows the generality of the above theorem. It is well known that for a standard Wiener process $\{W(t),\, t\geq 0\}$ one has (cf., e.g., (5.9) of Chapter X of Feller [121], or (2.5) in [**9**])

$$P\Big\{\sup_{0\leq s\leq a_T/2}|W(t+s)-W(t)|\leq x^{1/2}\Big\}\leq 2\exp\Big(-\frac{\pi^2a_T}{16x}\Big) \tag{3.74}$$

for any $t\geq 0$, $a_T>0$, $0<x\leq a_T$. Therefore, by (3.73)

$$\liminf_{T\to\infty}\inf_{0\leq t\leq b_T}\sup_{0\leq s\leq a_T}\frac{|W(t+s)-W(t)|}{(\pi^2a_T/32(\log(b_T/a_T)+\log\log\tilde{a}_T))^{1/2}}\geq\frac{1}{2}\quad \text{a.s.} \tag{3.75}$$

provided that (3.71) is satisfied. Such examples are, for example, provided by taking ($b_T=0$, $a_T=1/T$), ($b_T=1$, $a_T=1/T$), ($b_T=0$, $a_T=T$), ($b_T=T$), ($b_T=T$, $a_T=1$). For details we refer to page 32 of [**144**]. We note that in all of the above results, the thus obtained lim inf rates, up to a constant, are sharp (cf. CsR [A1, Chapter 1)]. For example, in the case of $(b_T = 1,\, a_T = \frac{1}{T} =: h)$, we get

$$\liminf_{h\downarrow 0} \inf_{0\le t\le 1} \sup_{0\le s\le h} 4\sqrt{\frac{8\log(1/h)}{\pi^2 h}}|W(t+s)-W(t)| \ge 1 \quad \text{a.s.}, \tag{3.76}$$

whose rate, in view of (2.75), is seen to be sharp.

Specific applications of the above quoted Theorem 2.1 of [**144**] include the case of $\Gamma(\cdot)$ being a *Banach space valued stochastic process with independent increments*, which in turn leads to studying $\Gamma(\cdot)$ being a *real valued stable process* with index $\alpha > 1$, or a *symmetric stable process* with index $0 < \alpha \le 1$. As another application of Theorem 2.1, the *limit inferior* problem is studied in general *for real valued Gaussian processes with mean zero and stationary increments.* Moreover, it is shown in particular, that the thus obtained lower bound is best possible for $X(t)$ of (3.65). As a consequence, it is also concluded that, under suitable conditions, almost all sample functions of the process $X(\cdot)$ of (3.65) are nowhere differentiable. Finally we note that almost all the known limit superior results on $X(\cdot)$, or ℓ^p–valued Gaussian processes (cf. [**130**], [**133**], [**138**], [**150**]) parallel the corresponding ones for the standard Wiener process (cf., e.g., Chapter 1 of CsR [A1]). On the other hand, the results of [**144**] show that the situation is quite different for the limit inferior behaviour of these processes. It appears that the *limit inferior behaviour* of such processes *is more sensitive to their deviations from a standard Wiener process than their limit superior behaviour.*

Path properties of kernel generated two-time parameter Gaussian processes. As already noted above, studying path properties of the two-time parameter Gaussian process $X(t,n)$ of (3.66) were initiated in [**114**] with Lin (cf. also Csáki et al. [**125**]). Integrating the equations in (3.64) from $-\infty$ to t, one obtains

$$X_k(t) = \int_{-\infty}^{t} \exp(-\lambda_k|t-s|)(2\gamma_k)^{1/2}dW_k(s), \quad k = 1,2,\ldots, \tag{3.77}$$

and hence also

$$X(t,n) = \sum_{k=1}^{n} X_k(t) = \sum_{k=1}^{n}\int_{-\infty}^{t} \exp(-\lambda_k|t-s|)(2\gamma_k)^{1/2}dW_k(s). \tag{3.78}$$

The latter, in turn, has in [**114**] led to studying also the two-time parameter Gaussian process

$$X(t,v) := \int_0^v \int_0^t \exp(-\lambda(y)(t-x))(2\gamma(y))^{1/2}dW(x,y), \tag{3.79}$$

where $\gamma(y)$, $\lambda(y)$ are assumed to be positive continuous functions on $[0,\infty)$, and $\{W(x,y),\, x \in \mathbf{R}^1,\, y \in \mathbf{R}^1_+\}$ is a standard Wiener sheet (cf., e.g., Sections 1.10–1.15 and the Supplementary remarks of Chapter 1 of CsR [A1]).

In view of (3.79), the papers [**124**] and [**139**] respectively with Lin, and Lin and Shao, study two-time parameter Gaussian processes $\{X(t,v),\, t\in\mathbf{R}^1,\, v\in\mathbf{R}^1_+\}$ of the form

$$X(t,v) := \int_0^\infty \int_{-\infty}^{\infty} \Gamma(t,v,x,y)dW(x,y), \tag{3.80}$$

where the kernel function $\Gamma(t,v,x,y)$ is assumed to be square integrable in (x,y). Thus $X(t,v)$ is a Gaussian process with mean zero and covariance function

$$\mathrm{Cov}(X(t,v)X(s,u)) = \int_0^\infty \int_{-\infty}^\infty \Gamma(t,v,x,y)\Gamma(s,u,x,y)dxdy. \tag{3.81}$$

No stationarity properties are assumed for $X(t,v)$, and it can be assumed to be separable. Varying the kernel Γ, special cases of this process include the Wiener sheet $X(t,v) = W(t,v)$, the Kiefer process $X(t,v) = W(t,v) - tW(1,v)$, and processes related to the infinite-dimensional, ℓ^2–valued Ornstein-Uhlenbeck process (cf. Examples 1–5 of [**139**]). Paper [**139**] improves the results of [**124**] to a great extent. In particular, the large deviation results of [**139**] are much sharper than those of [**124**], and are more like those of Fernique (cf., e.g., [122], [123], [124] and [125]). However, in [**139**] there is more attention being paid to details in the case of non-stationary Gaussian processes, especially when they are kernel generated as in (3.80). The hinted at large deviation results are used to establish fine path properties of the $X(\cdot,\cdot)$ process in general (cf. Theorems 1.1–1.9 of [**139**]) and those of the five examples of the paper in particular (cf. Corollaries 1.1–1.9 of [**139**]). First four of the just mentioned corollaries deal with the two-time parameter Wiener and Kiefer processes. The latter results are similar to those of Chapter 1 of CsR [A1] for the same processes. For further references and results on the Wiener sheet we refer to Khoshnevisan [171] in [V2]. The conclusions of Corollaries 1.5–1.9 were brand new at that time. Corollary 1.7 is related to some results of Walsh [294]. The discursive introduction of paper [**139**] is recommended as a particularly interesting reading on the history behind the problems in hand. For an extensive review and further references on path properties of Gaussian and related processes we refer to the research monographs [188], [189].

On the topics of some of the papers that are discussed in Csáki et al. [69] in [V2]. For an authoritative study of the contributions of the papers [**58**], [**108**], [**134**], [**152**], plus some more, on *Brownian local time and additive functionals* we refer to Sections 2.1–2.3 of Csáki et al. [69], for that of the papers [**178**], [**182**], [**187**] on *principal value of Brownian local time* to Section 2.4 of [69], and for that of [**176**] on *integral functionals of geometric stochastic processes* to Section 2.5 of [69] in [V2]. Section 3.1 of the same paper deals with *iterated processes* in view of parts of [**108**] and Burdzy [39] (cf. (2.8) and (3.2) respectively, in Csáki et al. [69]) via the *global Strassen-type functional* LIL *laws* of [**153**]. The papers [**156**], [**164**] study *local time* and *occupation time properties of iterated Brownian motion*. For details in this regard we refer to Section 3.2 of Csáki et al. [69]. Paper [**170**] with Zhan Shi and Marc Yor establishes *Strassen-type functional* LIL *joint laws* for the *normalized level crossings at zero* and the *maximal level crossings of the uniform empirical process* α_n (cf. (3.12)), and studies LIL *and other* LIL*–type laws for the local time of* α_n *under* L^p–norm as well (cf. Theorems 4.1–4.3 in Csáki et al. [69] in [V2]).

Strong limit theorems and invariance principles for self-normalized and Studentized partial sums processes in the domain of attraction of the normal law. The papers Csörgő et al. [**74**], [**75**] prompted [**97**] with Horváth, and [76] by S. Csörgő on *self-normalized sums from the domain of attraction of a stable law.* The Griffin and Kuelbs [135], [136] papers inspired the contributions Csörgő, Lin and Shao [**141**], and Csörgő and Shao [**143**] on *Studentized increments of partial sums*, and *self-normalized* Erdős–Rényi *type strong laws*, respectively, in view of CsR [**44**] and CsR [A1, Chapter 3]. The results of Darling and Erdős [85], Einmahl [113], Einmahl and Mason [114] prompted the Csörgő, Szyszkowicz and Wang

[**190**] exposition on establishing a *Darling–Erdős theorem for self-normalized partial sums processes* (cf. also Q. Wang [295] in [V2]). The Gine, Götze and Mason [130] *characterization* of the *asymptotic normality* of the *Student t–statistic in the domain of attraction of the normal law* [DAN] inspired the two expositions [**191**] and [**197**] by Csörgő, Szyszkowicz and Wang on *Donsker's theorem* and *weighted approximations for self-normalized and Studentized partial sums processes* in DAN. Paper [**192**] by the same three authors in [V2] revisits all these topics on self-normalized and Studentized partial sums processes. In the case of Donsker's theorem and weighted approximations for such partial sums in DAN, the relationship of these results to their scalar normalized sums companion duals in DAN is also explored (cf. Sections 2 and 3 of [**192**] in [V2]).

On reading paper [**192**], Yuliya Martsynyuk has called our attention to a number of significant developments concerning the domain of attraction of the multivariate normal law, the so-called generalized domain of attraction of the normal law, denoted here by GDAN, in view of the use of DAN in [**192**]. We conclude our discussion with recording some of her findings here in this regard with our many thanks to her. To begin with, the fundamental paper [139] by M.G. Hahn and M.J. Klass has played a seminal role in initiating an intensive study of GDAN by many authors. Considering stable distributions in R^2 via using componentwise norming, S. Resnick and P. Greenwood [231] establish an auxiliary result that is equivalent to the equivalence of (a) and (c) in Proposition 2.1 of [**192**] in [V2], which is a scalar normalized sums companion to Theorem 2.3 in the same paper. Prolific Studentization ideas are demonstrated via GDAN in R.A. Maller [193] and H.T.V. Vu et al. [293], where one can, in particular, find genuinely multivariate companions to the first conclusion of Theorem 2.6 in [**192**] that (a) implies (b). As to multivariate companions to the respective conclusions in both Proposition 2.1 and Theorem 2.6 of [**192**] that (a) implies (c), we refer to S.J. Sepanski [255]. We note in passing that, as far as we know, there is no converse yet to the multivariate version of (a) implies (b) in Theorem 2.3 of [**192**]. Hence the aforementioned results in [193], [293] and [255] are all one-sided, as opposed to the equivalences in paper [**192**] in [V2] on account of Giné et al. [130].

Lajos Horváth
Barbara Szyszkowicz

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