

# On the Csörgő-Révész increments of finite dimensional Gaussian random fields

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## Abstract

In this paper, we establish some limit theorems on the combined Csörgő-Révész increments with moduli of continuity for finite dimensional Gaussian random fields under mild conditions, via estimating upper bounds of large deviation probabilities on suprema of the finite dimensional Gaussian random fields.

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## 1. Introduction and results

Csörgő and Révész [8] proved some limit theorems on increments of the Wiener process  $W(t)$ ,  $0 \leq t < \infty$ , which are related to the well-known Erdős-Rényi law. Among their theorems, we introduce the following result:

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**Theorem A.** *Let  $a_T$  be a nondecreasing function of  $T > 0$  such that  $0 < a_T \leq T$ ,  $T/a_T$  is nondecreasing and  $\lim_{T \rightarrow \infty} (\log(T/a_T))/\log \log T = \infty$ . Then we have*

$$\lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T} \frac{|W(t + a_T) - W(t)|}{\sqrt{a_T} \gamma_T} = 1 \quad \text{a.s.}, \quad (1.1)$$

where  $\gamma_T = \left\{ 2 \left( \log(T/a_T) + \log \log T \right) \right\}^{1/2}$ ,  $T > e$ .

Since then, many various limit theories on the similar Csörgő-Révész type increments as in (1.1) for fractional Brownian motions, renewal processes, partial sum processes, Gaussian processes and related stochastic processes have been developed in [1, 3, 4, 6, 9, 10, 12, 13, 16, 18, 21, 26] and etc.

On the other hand, Csáki et al. [8] obtained the following modulus of continuity for a centered Gaussian process  $X(t)$  ( $0 \leq t < \infty$ ) with stationary increments  $\sigma^2(s) := E\{X(t+s) - X(t)\}^2$ :

**Theorem B.** *Let  $\sigma(s)$  be a nondecreasing and continuous regularly varying function with exponent  $\alpha$  ( $0 < \alpha < 1$ ) at 0, that is,  $\lim_{s \rightarrow 0} \{\sigma(xs)/\sigma(s)\} = x^\alpha$  for  $x > 0$ . Assume that, for any  $a < b < c < d$ ,*

$$E \left\{ \left( X(b) - X(a) \right) \left( X(d) - X(c) \right) \right\} \leq 0. \quad (1.2)$$

Then we have

$$\lim_{h \rightarrow 0} \sup_{0 \leq t \leq 1} \frac{|X(t+h) - X(t)|}{\sigma(h) \sqrt{2 \log(1/h)}} = 1 \quad \text{a.s.} \quad (1.3)$$

In this paper, we establish some limit theorems on the combined Csörgő-Révész increments with moduli of continuity for finite dimensional Gaussian random fields under mild conditions under mild conditions.

Throughout the paper, we always assume the following conditions: Let  $\{X_j(\mathbf{t}), \mathbf{t} \in [0, \infty)^N\}$ ,  $j = 1, 2, \dots, d$ , be real-valued continuous and centered Gaussian processes with  $X_j(\mathbf{0}) = 0$  and  $E\{X_j(\mathbf{t}) - X_j(\mathbf{s})\}^2 = \sigma_j^2(\|\mathbf{t} - \mathbf{s}\|)$ , where  $\sigma_j(h)$  are positive continuous functions of  $h > 0$  and  $\|\cdot\|$  is the usual Euclidean norm. Put

$$\sigma(d, h) = \max_{1 \leq j \leq d} \sigma_j(h)$$

and assume that, for some  $\alpha > 0$ ,  $\sigma(d, h)/h^\alpha$  is *quasi-increasing*, that is, there is a constant  $c > 0$  such that  $\sigma(d, s)/s^\alpha \leq c \sigma(d, t)/t^\alpha$  for  $0 < s < t < \infty$ .

Let  $\{X^d(\mathbf{t}) = (X_1(\mathbf{t}), \dots, X_d(\mathbf{t})), \mathbf{t} \in [0, \infty)^N\}$  be a  $d$ -dimensional Gaussian process with norm  $\|\cdot\|$  and  $N$  parameters  $t_1, \dots, t_N \in [0, \infty)$ , where  $\mathbf{t} = (t_1, \dots, t_N)$ . We call the process  $\{X^d(\mathbf{t}), \mathbf{t} \in [0, \infty)^N\}$  a  $d$ -dimensional *Gaussian random field*. The realizations of random fields  $\{X_j(\mathbf{t}), \mathbf{t} \in [0, \infty)^N\}$  for  $j = 1, 2, \dots, d$ , are assumed to be different objects. Moreover, the choice of coordinates of the parameter  $\mathbf{t} = (t_1, \dots, t_N)$  is not necessarily limited to length and time but any scale of measurement might be involved.

Now, we introduce some notations to be used in this paper: Let  $\mathbf{t} = (t_1, \dots, t_N)$  and  $\mathbf{s} = (s_1, \dots, s_N)$  be vectors in  $[0, \infty)^N$ . Denote:

$$\begin{aligned} \mathbf{0} &= (0, \dots, 0) \quad \text{and} \quad \mathbf{1} = (1, \dots, 1) \quad \text{in} \quad [0, \infty)^N, \\ \mathbf{t} \leq \mathbf{s} &\quad \text{if} \quad t_i \leq s_i \quad \text{for all integers} \quad 1 \leq i \leq N, \\ \mathbf{t} \pm \mathbf{s} &= (t_1 \pm s_1, \dots, t_N \pm s_N), \quad \mathbf{t}\mathbf{s} = (t_1 s_1, \dots, t_N s_N), \\ a\mathbf{t} &= (at_1, \dots, at_N) \quad \text{for} \quad a \in (-\infty, \infty), \\ \mathbf{h} &= (h_1, \dots, h_N) \in \left(0, \frac{1}{\sqrt{N}}\right)^N. \end{aligned}$$

For  $i = 1, \dots, N$ , let  $f_i(\mathbf{h})$  and  $g_i(\mathbf{h})$  be positive real-valued continuous functions. Define:

$$\begin{aligned} \mathbf{f}(\mathbf{h}) &= (f_1(\mathbf{h}), \dots, f_N(\mathbf{h})), \quad \mathbf{g}(\mathbf{h}) = (g_1(\mathbf{h}), \dots, g_N(\mathbf{h})), \\ \gamma_1(\mathbf{h}) &= \left\{ 2 \left( N \log \left( \frac{\|\mathbf{f}(\mathbf{h})\|}{\|\mathbf{g}(\mathbf{h})\|} \right) + \log |\log \sigma(d, \|\mathbf{g}(\mathbf{h})\|)| \right) \right\}^{1/2}, \\ \gamma_2(\mathbf{h}) &= \left\{ 2N \log \left( \frac{\|\mathbf{f}(\mathbf{h})\|}{\|\mathbf{g}(\mathbf{h})\|} \right) \right\}^{1/2}, \end{aligned}$$

where  $\log x = \ln(\max\{x, 1\})$ .

The following results generalize some main theorems on the Csörgő-Révész type increments of one dimensional Gaussian processes with one parameter by setting  $h = 1/T$  ( $0 < T < \infty$ ) in [1, 5, 8, 18, 26]. The main results are as follows:

**Theorem 1.1.** *Suppose that*

$$(i) \quad \|\mathbf{f}(\mathbf{h})/\mathbf{g}(\mathbf{h})\| + \|\mathbf{g}(\mathbf{h})\| \rightarrow \infty \quad \text{as } \mathbf{h} \rightarrow \mathbf{0}.$$

*Then we have*

$$\limsup_{\mathbf{h} \rightarrow \mathbf{0}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h})} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{g}(\mathbf{h})} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h})\|)\gamma_1(\mathbf{h})} \leq 1 \quad \text{a.s.} \quad (1.4)$$

The condition (i) implies that  $\mathbf{f}(\mathbf{h})$  and  $\mathbf{g}(\mathbf{h})$  may be many diverse functions. However, in order to obtain the opposite inequality of (1.4), the conditions on  $\mathbf{f}(\mathbf{h})$ ,  $\mathbf{g}(\mathbf{h})$  and  $\sigma(d, \cdot)$  are a little bit restricted as in the following Theorem 1.2:

**Theorem 1.2.** *Suppose that  $\sigma(d, h)$  is a regularly varying function with exponent  $\alpha$  ( $0 < \alpha < 1$ ) at 0 or  $\infty$  and that there exist positive constants  $c_1$  and  $c_2$  such that, for  $h > 0$ ,*

$$(ii) \quad \left| \frac{d\sigma^2(d, h)}{dh} \right| \leq c_1 \frac{\sigma^2(d, h)}{h} \quad \text{and} \quad \left| \frac{d^2\sigma^2(d, h)}{dh^2} \right| \leq c_2 \frac{\sigma^2(d, h)}{h^2}.$$

*Assume that*

$$(iii) \quad \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\log(\|\mathbf{f}(\mathbf{h})\|/\|\mathbf{g}(\mathbf{h})\|)}{\log|\log\|\mathbf{g}(\mathbf{h})\||} = \infty.$$

*Then we have*

$$\liminf_{\mathbf{h} \rightarrow \mathbf{0}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h})} \frac{\|X^d(\mathbf{t} + \mathbf{g}(\mathbf{h})) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h})\|)\gamma_2(\mathbf{h})} \geq 1 \quad \text{a.s.} \quad (1.5)$$

The class of variance functions  $\sigma^2$  satisfying (ii) contains all concave functions with  $0 < \alpha \leq 1/2$  (e.g.  $\sigma^2(d, h) = \sqrt{h}$ ) and convex functions with  $1/2 < \alpha < 1$ . We recall that the correlation function on increments of a stochastic process with stationary increments is nonpositive if and only if its variance function is nearly concave (cf. see (2.7) and (1.2) of this paper, (4.2) in Csáki et al. [5] and (2.7) in Lin and Qin [20]), and vice versa.

Also, the condition (iii) guarantees that the class of vector functions  $\mathbf{f}(\mathbf{h})$  and  $\mathbf{g}(\mathbf{h})$  contains many various functions such that  $\|\mathbf{f}(\mathbf{h})\|$  and  $\|\mathbf{g}(\mathbf{h})\|$ , respectively, can go to 0,  $\infty$  or constants as  $\mathbf{h}$  tends to  $\mathbf{0}$  (cf. Examples 1.1-1.3 of this paper).

From Theorems 1.1 and 1.2, we have the following

**Corollary 1.1.** *Under the assumptions of Theorem 1.2, we have*

$$\begin{aligned} \lim_{\mathbf{h} \rightarrow \mathbf{0}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h})} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{g}(\mathbf{h})} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h})\|)\gamma_2(\mathbf{h})} &= 1, \quad \text{a.s.}, \\ \lim_{\mathbf{h} \rightarrow \mathbf{0}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h})} \frac{\|X^d(\mathbf{t} + \mathbf{g}(\mathbf{h})) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h})\|)\gamma_2(\mathbf{h})} &= 1, \quad \text{a.s.} \end{aligned} \quad (1.6)$$

For one-parameter Wiener process with  $\sigma(1, h) = \sqrt{h}$ , the similar results as (1.6) can be found in Csörgő and Révész [8], Lin and Lu [18] and etc. With  $N = 1$  in Corollary 1.1, let  $f(h) = 1/h = T$  ( $h > 0$ ),  $\sigma(1, h) = \sqrt{h}$  and  $g(h) = a_T \leq T$  ( $a_T$  may be a constant). Then it is obvious that (1.1) follows from Corollary 1.1 under conditions of Theorem A.

The above results can be applied to develop the limit theories on the Csörgő-Révész increments of finite dimensional random fields with respect to the following stochastic processes: Ornstein-Uhlenbeck process (e.g. [5]), renewal process [24], lag sum process [2, 18], local-time process [6, 19], partial sum process [7, 12, 25], self-normalized partial sum process [11, 22] and etc.

**Example 1.1.** (large incremental result) Let  $\{X_j(\mathbf{t}), \mathbf{t} \in [0, \infty)^N\}$ ,  $j = 1, 2, \dots, d$ , be  $N$ -parameter fractional Brownian motions of orders  $\alpha_j$  with  $0 < \alpha_j < 1$ , that is, let  $\{X_j(\mathbf{t}), \mathbf{t} \in [0, \infty)^N\}$ ,  $j = 1, 2, \dots, d$ , be Gaussian random fields with  $X_j(\mathbf{0}) = 0$  and  $\sigma_j(h) = h^{\alpha_j}$ ,  $h > 0$ . When  $\alpha_j = 1/2$ , then  $\{X_j(\mathbf{t}), \mathbf{t} \in [0, \infty)^N\}$  are standard Wiener random fields. For each  $i = 1, 2, \dots, N$ , let  $h_i = 2^i e^{-T}$  for  $T > \log(2^N \sqrt{N})$ . Then  $\mathbf{h} = (h_1, \dots, h_N) = e^{-T}(2, \dots, 2^N)$ . For convenience, put

$$g_i(\mathbf{h}) = i \left( \prod_{k=1}^N h_k \right)^{-1} \quad \text{and} \quad \mathbf{f}(\mathbf{h}) = e^{NT} \mathbf{g}(\mathbf{h}).$$

Then  $\sigma_j(h)$ ,  $\mathbf{f}(\mathbf{h})$  and  $\mathbf{g}(\mathbf{h})$  satisfy all conditions of Corollary 1.1 with

$$\begin{aligned} \mathbf{g}(\mathbf{h}) &= e^{NT} 2^{-N(N+1)/2} (1, 2, \dots, N) =: \mathbf{G}_1(T) \rightarrow \infty \quad \text{as } T \rightarrow \infty, \\ \|\mathbf{g}(\mathbf{h})\| &= 2^{-N(N+1)/2} \left\{ N(N+1)(2N+1)/6 \right\}^{1/2} e^{NT} =: A_N e^{NT}, \\ \gamma_2(\mathbf{h}) &= N\sqrt{2T} \quad \text{and} \quad \sigma(d, \|\mathbf{g}(\mathbf{h})\|) = (A_N e^{NT})^\alpha \quad \text{for } \alpha = \max_{1 \leq j \leq d} \alpha_j. \end{aligned}$$

Thus we have, by Corollary 1.1,

$$\begin{aligned} \lim_{T \rightarrow \infty} \sup_{\mathbf{0} \leq \mathbf{t} \leq e^{NT} \mathbf{G}_1(T)} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{G}_1(T)} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{e^{\alpha NT} \sqrt{T}} &= \sqrt{2N} (A_N)^\alpha \quad \text{a.s.}, \\ \lim_{T \rightarrow \infty} \sup_{\mathbf{0} \leq \mathbf{t} \leq e^{NT} \mathbf{G}_1(T)} \frac{\|X^d(\mathbf{t} + \mathbf{G}_1(T)) - X^d(\mathbf{t})\|}{e^{\alpha NT} \sqrt{T}} &= \sqrt{2N} (A_N)^\alpha \quad \text{a.s.} \end{aligned} \quad (1.7)$$

**Example 1.2.** (small incremental result) Let  $\{X_j(\mathbf{t}), \mathbf{t} \in [0, \infty)^N\}$ ,  $j = 1, \dots, d$ , be as in Lemma 1.1. For each  $i = 1, 2, \dots, N$ , put  $h_i = \sqrt{i} e^{-T}$  for  $T > \log N$ . Define:

$$g_i(\mathbf{h}) = i \left( \prod_{k=1}^N h_k \right) \quad \text{and} \quad \mathbf{f}(\mathbf{h}) = e^{NT/2} \mathbf{g}(\mathbf{h}).$$

Then

$$\begin{aligned} \mathbf{g}(\mathbf{h}) &= \sqrt{N!} e^{-NT} (1, 2, \dots, N) =: \mathbf{G}_2(T) \rightarrow 0 \quad \text{as } T \rightarrow \infty, \\ \|\mathbf{g}(\mathbf{h})\| &= \left\{ (N+1)! N(2N+1)/6 \right\}^{1/2} e^{-NT} =: B_N e^{-NT}, \\ \gamma_2(\mathbf{h}) &= N\sqrt{T} \quad \text{and} \quad \sigma(d, \|\mathbf{g}(\mathbf{h})\|) = (B_N e^{-NT})^{\alpha'} \quad \text{for } \alpha' = \min_{1 \leq j \leq d} \alpha_j. \end{aligned}$$

Thus we have, by Corollary 1.1,

$$\begin{aligned} \lim_{T \rightarrow \infty} \sup_{\mathbf{0} \leq \mathbf{t} \leq e^{NT/2} \mathbf{G}_2(T)} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{G}_2(T)} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{e^{-\alpha' NT} \sqrt{T}} &= N(B_N)^{\alpha'} \quad \text{a.s.}, \\ \lim_{T \rightarrow \infty} \sup_{\mathbf{0} \leq \mathbf{t} \leq e^{NT/2} \mathbf{G}_2(T)} \frac{\|X^d(\mathbf{t} + \mathbf{G}_2(T)) - X^d(\mathbf{t})\|}{e^{-\alpha' NT} \sqrt{T}} &= N(B_N)^{\alpha'} \quad \text{a.s.} \end{aligned} \tag{1.8}$$

The following example is an extension of Theorem B (cf. [4]):

**Example 1.3.** (generalized modulus of continuity) Put  $\mathbf{f}(\mathbf{h}) = \mathbf{100}$  and  $\mathbf{g}(\mathbf{h}) = \mathbf{h}$  in Corollary 1.1. Then we have

$$\begin{aligned} \lim_{\mathbf{h} \rightarrow \mathbf{0}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{100}} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{h}} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{h}\|) \sqrt{\log(1/\|\mathbf{h}\|)}} &= \sqrt{2N} \quad \text{a.s.}, \\ \lim_{\mathbf{h} \rightarrow \mathbf{0}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{100}} \frac{\|X^d(\mathbf{t} + \mathbf{h}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{h}\|) \sqrt{\log(1/\|\mathbf{h}\|)}} &= \sqrt{2N} \quad \text{a.s.} \end{aligned} \tag{1.9}$$

## 2. Proof of main results

In order to prove Theorem 1.1, we need the following lemma (cf. Lin and Choi [17]):

**Lemma 2.1.** For any  $\varepsilon > 0$  there exists a positive constant  $C_\varepsilon$  depending only on  $\varepsilon$  such that, for all  $x > 1$ ,

$$P \left\{ \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h})} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{g}(\mathbf{h})} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h})\|)} \geq x \right\} \\ \leq C_\varepsilon \left( \frac{\|\mathbf{f}(\mathbf{h})\|}{\|\mathbf{g}(\mathbf{h})\|} \right)^N x^{d-2} \exp \left( - \frac{2x^2}{(2 + \varepsilon)^2} \right).$$

**Proof of Theorem 1.1.** Let  $\theta = 1 + \varepsilon$  for any given  $\varepsilon > 0$ . Define

$$A_k = \{ \mathbf{h} : \theta^k \leq \frac{\|\mathbf{f}(\mathbf{h})\|}{\|\mathbf{g}(\mathbf{h})\|} \leq \theta^{k+1} \}, \quad 0 \leq k < \infty, \\ A_{k,j} = \{ \mathbf{h} : \theta^j \leq \sigma(d, \|\mathbf{g}(\mathbf{h})\|) \leq \theta^{j+1}, \mathbf{h} \in A_k \}, \quad -\infty < j < \infty, \\ g_i(\mathbf{h}_{k,j}) = \sup \{ g_i(\mathbf{h}) : \mathbf{h} \in A_{k,j} \}, \quad i = 1, \dots, N, \\ f_i(\mathbf{h}_{k,j}) = \sup \{ f_i(\mathbf{h}) : \mathbf{h} \in A_{k,j} \}, \quad i = 1, \dots, N.$$

By the condition (i), we have

$$\limsup_{\mathbf{h} \rightarrow \mathbf{0}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h})} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{g}(\mathbf{h})} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h})\|) \gamma_1(\mathbf{h})} \\ \leq \limsup_{|j|+l \rightarrow \infty} \sup_{k \geq l \geq 0} \sup_{\mathbf{h} \in A_{k,j}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h})} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{g}(\mathbf{h})} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h})\|) \gamma_1(\mathbf{h})} \\ \leq \limsup_{|j|+l \rightarrow \infty} \sup_{k \geq l} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h}_{k,j})} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{g}(\mathbf{h}_{k,j})} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\theta^j D(k, j)} \\ \leq \theta \limsup_{|j|+l \rightarrow \infty} \sup_{k \geq l} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h}_{k,j})} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{g}(\mathbf{h}_{k,j})} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h}_{k,j})\|) D(k, j)}. \quad (2.1)$$

where  $D(k, j) = \{2(\log \theta^{kN} + \log \log \theta^{|j|})\}^{1/2}$ . Now, we will show that

$$\limsup_{|j|+l \rightarrow \infty} \sup_{k \geq l} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h}_{k,j})} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{g}(\mathbf{h}_{k,j})} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h}_{k,j})\|) D(k, j)} \leq \theta \quad \text{a.s.} \quad (2.2)$$

Applying Lemma 2.1, there exists  $C_\varepsilon > 0$  such that

$$\begin{aligned}
& P \left\{ \sup_{k \geq l} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h}_{k,j})} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{g}(\mathbf{h}_{k,j})} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h}_{k,j})\|)} > \theta D(k, j) \right\} \\
& \leq C_\varepsilon \sum_{k \geq 1} \left( \frac{\|\mathbf{f}(\mathbf{h}_{k,j})\|}{\|\mathbf{g}(\mathbf{h}_{k,j})\|} \right)^N \exp \left( - \frac{4(1 + \varepsilon)^2}{(2 + \varepsilon)^2} (\log \theta^{kN} + \log \log \theta^{|j|}) \right) \\
& \leq C_\varepsilon \sum_{k \geq l} \theta^{-\varepsilon kN/3} |j \vee 1|^{-(3+\varepsilon)/3} \\
& \leq C_\varepsilon |j \vee 1|^{-(3+\varepsilon)/3} \theta^{-\varepsilon lN/3}.
\end{aligned}$$

for sufficiently large  $|j| + l$ , where  $j \vee 1 = \max\{j, 1\}$ . Hence we have

$$\sum_{|j|=1}^{\infty} \sum_{l=0}^{\infty} P \left\{ \sup_{k \geq l} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h}_{k,j})} \sup_{\mathbf{0} \leq \mathbf{s} \leq \mathbf{g}(\mathbf{h}_{k,j})} \frac{\|X^d(\mathbf{t} + \mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h}_{k,j})\|)} D(k, j) > \theta \right\} < \infty,$$

and (2.2) follows from the Borel-Cantelli lemma. Combining (2.2) with (2.1) yields (1.1) by the arbitrariness of  $\theta$ . This completes the proof.  $\square$

The following Lemmas 2.2-2.5 are needed to prove Theorem 1.2:

**Lemma 2.2.** *Assume that the condition (ii) of Theorem 1.2 is satisfied. Let  $\mathbf{a} > \mathbf{0}$  and  $\mathbf{b} > \mathbf{1}$  be  $N$ -dimensional vectors. Then there exists a positive constant  $c$  such that*

$$\left| \int_{\|\mathbf{a}\| \|\mathbf{b}\|}^{\|\mathbf{a}\| \|\mathbf{b} + \mathbf{1}\|} d\sigma^2(d, x) - \int_{\|\mathbf{a}\| \|\mathbf{b} - \mathbf{1}\|}^{\|\mathbf{a}\| \|\mathbf{b}\|} d\sigma^2(d, x) \right| \leq c \frac{\sigma^2(d, \|\mathbf{a}\| \|\mathbf{b} + \mathbf{1}\|)}{\|\mathbf{b} - \mathbf{1}\|^2}.$$

**Proof.** We have

$$\begin{aligned}
& \left| \int_{\|\mathbf{a}\| \|\mathbf{b}\|}^{\|\mathbf{a}\| \|\mathbf{b} + \mathbf{1}\|} d\sigma^2(d, x) - \int_{\|\mathbf{a}\| \|\mathbf{b} - \mathbf{1}\|}^{\|\mathbf{a}\| \|\mathbf{b}\|} d\sigma^2(d, x) \right| \\
& = \left| \int_{\|\mathbf{a}\| \|\mathbf{b} - \mathbf{1}\|}^{\|\mathbf{a}\| \|\mathbf{b} + \mathbf{1}\| + \|\mathbf{a}\| \|\mathbf{b} - \mathbf{1}\| - \|\mathbf{a}\| \|\mathbf{b}\|} \left( \frac{d\sigma^2(d, x + \|\mathbf{a}\| \|\mathbf{b}\| - \|\mathbf{a}\| \|\mathbf{b} - \mathbf{1}\|)}{dx} \right. \right. \\
& \quad \left. \left. - \frac{d\sigma^2(d, x)}{dx} \right) dx + \int_{\|\mathbf{a}\| \|\mathbf{b}\|}^{\|\mathbf{a}\| \|\mathbf{b} + \mathbf{1}\| + \|\mathbf{a}\| \|\mathbf{b} - \mathbf{1}\| - \|\mathbf{a}\| \|\mathbf{b}\|} \left( \frac{d\sigma^2(d, x)}{dx} \right) dx \right| \\
& \leq \int_{\|\mathbf{a}\| \|\mathbf{b} - \mathbf{1}\|}^{\|\mathbf{a}\| \|\mathbf{b} + \mathbf{1}\| + \|\mathbf{a}\| \|\mathbf{b} - \mathbf{1}\| - \|\mathbf{a}\| \|\mathbf{b}\|} \int_x^{x + \|\mathbf{a}\| \|\mathbf{b}\| - \|\mathbf{a}\| \|\mathbf{b} - \mathbf{1}\|} \left| \frac{d^2\sigma^2(d, y)}{dy^2} \right| dy dx \\
& \quad + \int_{\|\mathbf{a}\| \|\mathbf{b}\|}^{\|\mathbf{a}\| \|\mathbf{b} + \mathbf{1}\| + \|\mathbf{a}\| \|\mathbf{b} - \mathbf{1}\| - \|\mathbf{a}\| \|\mathbf{b}\|} \left| \frac{d\sigma^2(d, x)}{dx} \right| dx \\
& =: I_1 + I_2, \quad \text{say.}
\end{aligned}$$



Thus

$$\begin{aligned}
I_1 &\leq \int_{\frac{\|\mathbf{a}\| \|\mathbf{b}-\mathbf{1}\|}{\|\mathbf{a}\| \|\mathbf{b}+\mathbf{1}\| + \|\mathbf{a}\| \|\mathbf{b}-\mathbf{1}\| - \|\mathbf{a}\| \|\mathbf{b}\|}}^{x + \|\mathbf{a}\| \|\mathbf{b}\| - \|\mathbf{a}\| \|\mathbf{b}-\mathbf{1}\|} \left( c_2 \frac{\sigma^2(d, y)}{y^2} \right) dy dx \\
&\leq c_2 \int_{\frac{\|\mathbf{a}\| \|\mathbf{b}-\mathbf{1}\|}{\|\mathbf{a}\| \|\mathbf{b}+\mathbf{1}\| + \|\mathbf{a}\| \|\mathbf{b}-\mathbf{1}\| - \|\mathbf{a}\| \|\mathbf{b}\|}} \left( \frac{\sigma^2(d, x + \|\mathbf{a}\| \|\mathbf{b}\| - \|\mathbf{a}\| \|\mathbf{b}-\mathbf{1}\|)}{x^2} \right) \\
&\quad \times (\|\mathbf{a}\| \|\mathbf{b}\| - \|\mathbf{a}\| \|\mathbf{b}-\mathbf{1}\|) dx \\
&\leq c_2 \frac{\sigma^2(d, \|\mathbf{a}\| \|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{a}\|^2 \|\mathbf{b}-\mathbf{1}\|^2} (\|\mathbf{a}\| \|\mathbf{b}+\mathbf{1}\| - \|\mathbf{a}\| \|\mathbf{b}\|) (\|\mathbf{a}\| \|\mathbf{b}\| - \|\mathbf{a}\| \|\mathbf{b}-\mathbf{1}\|) \\
&\leq c \frac{\sigma^2(d, \|\mathbf{a}\| \|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{b}-\mathbf{1}\|^2},
\end{aligned}$$

where  $c > 0$  is a constant, and

$$\begin{aligned}
I_2 &\leq \int_{\frac{\|\mathbf{a}\| \|\mathbf{b}\|}{\|\mathbf{a}\| \|\mathbf{b}+\mathbf{1}\| + \|\mathbf{a}\| \|\mathbf{b}-\mathbf{1}\| - \|\mathbf{a}\| \|\mathbf{b}\|}} \left( c_1 \frac{\sigma^2(d, x)}{x} \right) dx \\
&\leq c_1 \frac{\sigma^2(d, \|\mathbf{a}\| \|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{a}\| \|\mathbf{b}\|} (\|\mathbf{a}\| \|\mathbf{b}+\mathbf{1}\| - \|\mathbf{a}\| \|\mathbf{b}\| - (\|\mathbf{a}\| \|\mathbf{b}\| - \|\mathbf{a}\| \|\mathbf{b}-\mathbf{1}\|)) \\
&\leq c_1 \frac{\sigma^2(d, \|\mathbf{a}\| \|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{a}\| \|\mathbf{b}\|} \\
&\quad \times \left( \frac{\|\mathbf{a}\|^2 \|\mathbf{b}+\mathbf{1}\|^2 - \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - \|\mathbf{a}\|^2 \|\mathbf{b}-\mathbf{1}\|^2)}{2\|\mathbf{a}\| \|\mathbf{b}\|} \right) \\
&\leq c \frac{\sigma^2(d, \|\mathbf{a}\| \|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{b}-\mathbf{1}\|^2}. \quad \square
\end{aligned}$$

**Lemma 2.3.** [23] *Suppose that  $\{V_i, i = 1, \dots, n\}$  and  $\{W_i, i = 1, \dots, n\}$  are jointly standardized normal random variable with  $\text{Cov}(V_i, V_j) \leq \text{Cov}(W_i, W_j)$ ,  $i \neq j$ . Then, for any real  $u_i$  ( $i = 1, \dots, n$ ), we have*

$$P\{V_i \leq u_i, i = 1, \dots, n\} \leq P\{W_i \leq u_i, i = 1, \dots, n\},$$

$$P\{V_i \geq u_i, i = 1, \dots, n\} \leq P\{W_i \geq u_i, i = 1, \dots, n\}.$$

**Lemma 2.4.** [1, 3, 14, 15] *Let  $\mathbb{N} = (n_1, \dots, n_N)$  be a  $N$ -dimensional vector, where  $n_1, \dots, n_N = 1, 2, \dots, L$ . Suppose that  $\{Y(\mathbb{N})\}$  is a sequence of  $N$ -parameter standard normal random variables with  $\Lambda(\mathbb{N}, \mathbb{N}') := \text{Cov}(Y(\mathbb{N}), Y(\mathbb{N}'))$  for  $\mathbb{N} \neq \mathbb{N}'$  such that*

$$\delta := \max_{\mathbb{N} \neq \mathbb{N}'} |\Lambda(\mathbb{N}, \mathbb{N}')| < 1 \quad \text{and} \quad |\lambda(\mathbb{N}, \mathbb{N}')| := |\Lambda(l_{\mathbb{N}}, l_{\mathbb{N}'})| < \|\mathbb{N} - \mathbb{N}'\|^{-\nu}$$

for some  $\nu > 0$ , where  $\{l_{\mathbb{N}} = (l_{n_1}, \dots, l_{n_N})\}$  is a subsequence of  $\{\mathbb{N}\}$ . Denote  $\mathbf{m} = (m_1, \dots, m_N)$  with  $m_i \leq L$ ,  $1 \leq i \leq N$ . Set  $u = \{(2 - \eta) \log(\prod_{i=1}^N m_i)\}^{1/2}$ , where  $0 < \eta < (1 - \delta)\nu/(1 + \nu + \delta)$ . Then we have

$$P\left\{\max_{\mathbf{1} \leq \mathbb{N} \leq \mathbf{m}} Y(l_{\mathbb{N}}) \leq u\right\} \leq \{\Phi(u)\}^{\prod_{i=1}^N m_i} + c \left(\prod_{i=1}^N m_i\right)^{-\delta_0}, \quad (2.3)$$

where  $\Phi(u) = \int_{-\infty}^u \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$ ,  $\delta_0 = \{\nu(1 - \delta) - \eta(1 + \delta + \nu)\}/\{(1 + \nu)(1 + \delta)\} > 0$  and  $c > 0$  is a constant independent of  $\mathbb{N}$  and  $u$ .

**Proof of Theorem 1.2.** Let  $k_1, \dots, k_N, j_1, \dots, j_N$  be integers. Denote:

$$\mathbf{k} = (k_1, \dots, k_N), \quad \mathbf{j} = (j_1, \dots, j_N), \quad k = \frac{1}{N} \sum_{i=1}^N k_i, \quad j = \frac{1}{N} \sum_{i=1}^N j_i,$$

$$\Theta^{a\mathbf{j}} = (\theta^{aj_1}, \dots, \theta^{aj_N}) \quad \text{for any given } \theta > 1 \text{ and } a \in (-\infty, \infty).$$

Set

$$B_{\mathbf{k}, \mathbf{j}} = \{\mathbf{h} : \theta^{k_i - 1} \leq f_i(\mathbf{h}) \leq \theta^{k_i}, \quad \theta^{j_i - 1} \leq g_i(\mathbf{h}) \leq \theta^{j_i}, \quad 1 \leq i \leq N\}.$$

Note that  $\|\mathbf{g}(\mathbf{h})\| \geq \theta^{j-1}$  for  $\mathbf{h} \in B_{\mathbf{k}, \mathbf{j}}$ .

First, assume that  $\|\mathbf{g}(\mathbf{h})\| \rightarrow 0$  (or  $\infty$ ) as  $\mathbf{h} \rightarrow \mathbf{0}$ . By the condition (iii), there exists  $\gamma > 0$  such that

$$q := k - j > \gamma (\log \log \theta^{|j|}) / (\log \theta)^2 =: J$$

for sufficiently large  $|j|$ . Put  $m_i = [\theta^{k_i - j_i - 1}/M]$ ,  $1 \leq i \leq N$ , where  $M > 0$  is large

enough and  $[\cdot]$  denotes the integer part. By (iii), we can write

$$\begin{aligned}
& \liminf_{\mathbf{h} \rightarrow \mathbf{0}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h})} \frac{\|X^d(\mathbf{t} + \mathbf{g}(\mathbf{h})) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h})\|)\gamma_2(\mathbf{h})} \\
& \geq \liminf_{|j| \rightarrow \infty} \inf_{q > J} \sup_{\mathbf{0} \leq \mathbf{t} \leq \Theta^{k-1}} \frac{\|X^d(\mathbf{t} + \Theta^{\mathbf{j}}) - X^d(\mathbf{t})\|}{\sigma(d, \|\Theta^{\mathbf{j}}\|)(2 \log \prod_{i=1}^N m_i)^{1/2}} \\
& \quad - \limsup_{|j| \rightarrow \infty} \sup_{q > J} \sup_{\mathbf{0} \leq \mathbf{t} \leq \Theta^k} \sup_{\Theta^{j-1} \leq \mathbf{s} \leq \Theta^{\mathbf{j}}} \\
& \quad \frac{\|X^d(\mathbf{t} + \Theta^{\mathbf{j}}) - X^d(\mathbf{t} + \mathbf{s})\|}{\sigma(d, \|\Theta^{\mathbf{j}} - \Theta^{j-1}\|)(2 \log \prod_{i=1}^N m_i)^{1/2}} \frac{\sigma(d, \|\Theta^{\mathbf{j}} - \Theta^{j-1}\|)}{\sigma(d, \|\Theta^{j-1}\|)} \\
& =: Q_1 - Q_2.
\end{aligned} \tag{2.4}$$

First we will show that

$$Q_1 \geq 1 \quad \text{a.s.} \tag{2.5}$$

By the definition of  $\sigma(d, h)$ , there exists  $i_0 > 0$  ( $1 \leq i_0 \leq d$ ) such that  $\sigma_{i_0}(\|\Theta^{\mathbf{j}}\|) = \sigma(d, \|\Theta^{\mathbf{j}}\|)$ , where  $i_0 = i_0(\mathbf{j})$  depends on  $\mathbf{j}$ . Put  $\mathbf{m} = (m_1, \dots, m_N)$ . Then

$$\begin{aligned}
Q_1 & \geq \liminf_{|j| \rightarrow \infty} \inf_{q > J} \max_{\mathbf{1} \leq \mathbf{l} \leq \mathbf{m}} \frac{X_{i_0}((M\mathbf{l} + \mathbf{1})\Theta^{\mathbf{j}}) - X_{i_0}(M\mathbf{l}\Theta^{\mathbf{j}})}{\sigma_{i_0}(\|\Theta^{\mathbf{j}}\|)(2 \log \prod_{i=1}^N m_i)^{1/2}} \\
& =: \liminf_{|j| \rightarrow \infty} \inf_{q > J} \max_{\mathbf{1} \leq \mathbf{l} \leq \mathbf{m}} \frac{U_{\mathbf{j}}(\mathbf{l})}{(2 \log \prod_{i=1}^N m_i)^{1/2}}
\end{aligned} \tag{2.6}$$

and (2.5) is proved if we show that with probability one the right-hand side of (2.6) is greater than or equal to one. Using the elementary relation  $ab = (a^2 + b^2 - (a-b)^2)/2$ , then it follows that, for all  $\mathbf{l}$  and  $\mathbf{l}'$  with  $\mathbf{l} > \mathbf{l}'$ ,

$$\begin{aligned}
|\lambda_{\mathbf{j}}(\mathbf{l}, \mathbf{l}')| & := |\text{Cov}(U_{\mathbf{j}}(\mathbf{l}), U_{\mathbf{j}}(\mathbf{l}'))| \\
& = \frac{1}{2\sigma_{i_0}^2(\|\Theta^{\mathbf{j}}\|)} \left\{ \sigma_{i_0}^2(\|M(\mathbf{l} - \mathbf{l}')\Theta^{\mathbf{j}} + \Theta^{\mathbf{j}}\|) - \sigma_{i_0}^2(\|M(\mathbf{l} - \mathbf{l}')\Theta^{\mathbf{j}}\|) \right. \\
& \quad \left. - (\sigma_{i_0}^2(\|M(\mathbf{l} - \mathbf{l}')\Theta^{\mathbf{j}}\|) - \sigma_{i_0}^2(\|M(\mathbf{l} - \mathbf{l}')\Theta^{\mathbf{j}} - \Theta^{\mathbf{j}}\|)) \right\}.
\end{aligned} \tag{2.7}$$

If the right hand side of (2.7) is less than or equal to zero, then it follows from Lemma 2.3 that, for any  $0 < \varepsilon < 1$ ,

$$\begin{aligned}
& P \left\{ \inf_{q > J} \max_{\mathbf{1} \leq \mathbf{l} \leq \mathbf{m}} \frac{U_{\mathbf{j}}(\mathbf{l})}{\sqrt{2 \log \prod_{i=1}^N m_i}} \leq \sqrt{1 - \varepsilon} \right\} \\
& \leq \sum_{q > J} \left\{ \Phi \left( \sqrt{(2 - 2\varepsilon) \log \prod_{i=1}^N m_i} \right) \right\}^{\prod_{i=1}^N m_i}.
\end{aligned} \tag{2.8}$$

On the other hand, if the right hand side of (2.7) is positive, that is,  $\sigma_{i_0}^2$  is a nearly convex function, then it follows from the regular variation of  $\sigma_{i_0}^2$  and Lemma 2.2 with  $\mathbf{a} = \Theta^{\mathbf{j}}$  and  $\mathbf{b} = M(\mathbf{l} - \mathbf{l}')$  that

$$\begin{aligned}
|\lambda_{\mathbf{j}}(\mathbf{l}, \mathbf{l}')| &\leq \frac{1}{\sigma_{i_0}^2(\|\Theta^{\mathbf{j}}\|)} \left| \int_{\|\Theta^{\mathbf{j}}\| \|M(\mathbf{l}-\mathbf{l}')\|}^{\|\Theta^{\mathbf{j}}\| \|M(\mathbf{l}-\mathbf{l}')+\mathbf{1}\|} d\sigma_{i_0}^2(x) - \int_{\|\Theta^{\mathbf{j}}\| \|M(\mathbf{l}-\mathbf{l}')-\mathbf{1}\|}^{\|\Theta^{\mathbf{j}}\| \|M(\mathbf{l}-\mathbf{l}')\|} d\sigma_{i_0}^2(x) \right| \\
&\leq c \frac{\sigma_{i_0}^2(\|\Theta^{\mathbf{j}}\| \|M(\mathbf{l}-\mathbf{l}') + \mathbf{1}\|)}{\sigma_{i_0}^2(\|\Theta^{\mathbf{j}}\| \|M(\mathbf{l}-\mathbf{l}') - \mathbf{1}\|^2)} \\
&\leq c \frac{\|M(\mathbf{l}-\mathbf{l}') + \mathbf{1}\|^2}{\|M(\mathbf{l}-\mathbf{l}') - \mathbf{1}\|^2} \|M(\mathbf{l}-\mathbf{l}') + \mathbf{1}\|^{2\alpha-2} \\
&< \xi \|\mathbf{l} - \mathbf{l}'\|^{-\nu}
\end{aligned}$$

for sufficiently small  $\xi > 0$ , where  $\nu = 1 - \alpha > 0$ . Let us apply Lemma 2.4 for

$$\begin{aligned}
Y(\mathbf{l}) &= U_{\mathbf{j}}(\mathbf{l}), \quad \mathbf{1} \leq \mathbf{l} \leq \mathbf{m}, \\
|\lambda(\mathbf{l}, \mathbf{l}')| &= |\lambda_{\mathbf{j}}(\mathbf{l}, \mathbf{l}')| < \xi \|\mathbf{l} - \mathbf{l}'\|^{-\nu}, \\
u &= \{(2 - \eta) \log \prod_{i=1}^N m_i\}^{1/2}, \quad \eta = 2\varepsilon.
\end{aligned}$$

Then we have

$$\begin{aligned}
&P \left\{ \inf_{q>J} \max_{\mathbf{1} \leq \mathbf{l} \leq \mathbf{m}} \frac{U_{\mathbf{j}}(\mathbf{l})}{\sqrt{2 \log \prod_{i=1}^N m_i}} \leq \sqrt{1 - \varepsilon} \right\} \\
&\leq \sum_{q>J} \left\{ (\Phi(u))^{\prod_{i=1}^N m_i} + c \left( \prod_{i=1}^N m_i \right)^{-\delta_0} \right\} \\
&\leq \sum_{q>J} \left\{ \exp(-c\theta^{\varepsilon Nq}) + c(\theta^{Nq})^{-\delta_0} \right\} \tag{2.9} \\
&\leq c \sum_{q>J} \theta^{-N\delta_0 q} \leq c \theta^{-N\delta_0 \gamma (\log_{\theta} \log \theta^{|j|}) / \log \theta} \\
&\leq c |j|^{-N\delta_0 \gamma / \log \theta}
\end{aligned}$$

for sufficiently large  $|j|$ . Note that the right hand side of (2.8) is less than or equal to that of (2.9). Taking  $\theta > 1$  such that  $\log \theta < N\delta_0 \gamma$  in (2.9), then the Borel-Cantelli lemma implies (2.5) via (2.6). Now, we turn to show that

$$Q_2 \leq 2c\varepsilon^{\alpha/2} \quad \text{a.s.} \tag{2.10}$$

for any small  $\varepsilon > 0$ , where  $c > 0$  is a constant. Since  $\sigma(d, h)$  is regularly varying, we have

$$\frac{\sigma(d, \|\Theta^{\mathbf{j}} - \Theta^{\mathbf{j}-1}\|)}{\sigma(d, \|\Theta^{\mathbf{j}-1}\|)} \leq c\varepsilon^{\alpha/2}.$$

Therefore, (2.10) is proved if we show that

$$\limsup_{|j| \rightarrow \infty} \sup_{q > J} \sup_{\mathbf{0} \leq \mathbf{t} \leq \Theta^{\mathbf{k}}} \sup_{\Theta^{\mathbf{j}-1} \leq \mathbf{s} \leq \Theta^{\mathbf{j}}} \frac{\|X^d(\mathbf{t} + \Theta^{\mathbf{j}}) - X^d(\mathbf{t} + \mathbf{s})\|}{\sigma(d, \|\Theta^{\mathbf{j}} - \Theta^{\mathbf{j}-1}\|) \sqrt{2 \log \prod_{i=1}^N m_i}} \leq 2 \quad \text{a.s.} \quad (2.11)$$

Applying the same way as the proof of Lemma 2.1, then it follows that, for sufficiently large  $|j|$ ,

$$\begin{aligned} P \left\{ \sup_{\mathbf{0} \leq \mathbf{t} \leq \Theta^{\mathbf{k}}} \sup_{\Theta^{\mathbf{j}-1} \leq \mathbf{s} \leq \Theta^{\mathbf{j}}} \frac{\|X^d(\mathbf{t} + \Theta^{\mathbf{j}}) - X^d(\mathbf{t} + \mathbf{s})\|}{\sigma(d, \|\Theta^{\mathbf{j}} - \Theta^{\mathbf{j}-1}\|) \sqrt{2 \log \prod_{i=1}^N m_i}} \geq 2 + \varepsilon \right\} \\ \leq c \frac{\theta^{Nk}}{\|\Theta^{\mathbf{j}} - \Theta^{\mathbf{j}-1}\|^N} \exp \left( - \frac{4(2 + \varepsilon)^2}{(2 + \varepsilon)^2} Nq \log \theta \right) \\ \leq c \theta^{-3Nq}. \end{aligned}$$

Since

$$\sum_{|j|=1}^{\infty} \sum_{q > J} \theta^{-3Nq} \leq c \sum_{|j|=1}^{\infty} |j|^{-\gamma/\log \theta} < \infty,$$

we obtain (2.11) and hence (1.5) holds true by (2.10), (2.5) and (2.4).

Next, assume that the vector function  $\mathbf{g}$  is constant. For a constant  $c$  ( $-\infty < c < \infty$ ), let  $\mathbf{g}(\mathbf{h}) = \Theta^c = (\theta^c, \dots, \theta^c)$  be a  $N$ -dimensional vector. According to same lines as above, put  $m_i = \lceil \theta^{k_i - c - 1} / M \rceil$ ,  $1 \leq i \leq N$ , in  $B_{\mathbf{k}, \mathbf{j}}$ . By (iii), we can write

$$\begin{aligned} \liminf_{\mathbf{h} \rightarrow \mathbf{0}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{f}(\mathbf{h})} \frac{\|X^d(\mathbf{t} + \mathbf{g}(\mathbf{h})) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{g}(\mathbf{h})\|) \gamma_2(\mathbf{h})} \\ \geq \liminf_{k \rightarrow \infty} \sup_{\mathbf{0} \leq \mathbf{t} \leq \Theta^{\mathbf{k}-1}} \frac{\|X^d(\mathbf{t} + \Theta^c) - X^d(\mathbf{t})\|}{\sigma(d, \|\Theta^c\|) (2 \log \prod_{i=1}^N m_i)^{1/2}} \\ \geq \liminf_{k \rightarrow \infty} \max_{1 \leq l \leq \mathbf{m}} \frac{X_{i_0}((M\mathbf{l} + \mathbf{1})\Theta^c) - X_{i_0}(M\mathbf{l}\Theta^c)}{\sigma_{i_0}(\|\Theta^c\|) (2 \log \prod_{i=1}^N m_i)^{1/2}}. \end{aligned}$$

Defining

$$U_{\mathbf{c}}(\mathbf{l}) = \frac{X_{i_0}((M\mathbf{l} + \mathbf{1})\Theta^c) - X_{i_0}(M\mathbf{l}\Theta^c)}{\sigma_{i_0}(\|\Theta^c\|)},$$

we can obtain

$$P \left\{ \max_{1 \leq l \leq \mathbf{m}} \frac{U_{\mathbf{c}}(\mathbf{l})}{\sqrt{2 \log \prod_{i=1}^N m_i}} \leq \sqrt{1 - \varepsilon} \right\} \leq c_0 \theta^{-N \delta_0 k}$$

for some  $c_0 > 0$ . Thus

$$\sum_{k=1}^{\infty} P \left\{ \max_{1 \leq l \leq \mathbf{m}} \frac{U_{\mathbf{c}}(\mathbf{l})}{\sqrt{2 \log \prod_{i=1}^N m_i}} \leq \sqrt{1 - \varepsilon} \right\} < \infty.$$

This completes the proof of Theorem 1.2.  $\square$

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