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Path properties of (N, d)-Gaussian random fields

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Abstract

In this paper, we investigate several sample path properties on the increments of (N, d)-Gaussian random fields and also we obtain the law of iterated logarithm for the Gaussian random field, via estimating upper and lower bounds of large deviation probabilities on suprema of the (N, d)-Gaussian random fields.

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1. Introduction and results

The limit theory on the increments of Wiener processes, partial sum processes, empirical processes and etc. is initially integrated in Csörgő and Révész (1981).

Since then, many various limit theories for fractional Brownian motions, renewal processes, Gaussian processes and related stochastic processes have been developed in Csaki et al. (1991), Choi (1991), Csörgő and Shao ((1993), (1994)), Kôno (1996),

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Zhang ((1996), (1997)), Steinebach ((1983), (1998)), Choi and Kôno (1999), Lin and Choi ((1999), (2001)) and etc.

In this paper, we investigate several sample path properties on the increments of (N,d)-Gaussian random fields under mild conditions and also we prove the law of iterated logarithm for the Gaussian random field. Throughout the paper, we always assume the following conditions: Let $\{X_j(\mathbf{t}),\ \mathbf{t}\in[0,\infty)^N\}$, $j=1,2,\cdots,d$, be real-valued continuous and centered Gaussian processes with $X_j(\mathbf{0})=0$ and $E\{X_j(\mathbf{t})-X_j(\mathbf{s})\}^2=\sigma_j^2(\|\mathbf{t}-\mathbf{s}\|)$, where $\sigma_j(h)$ are positive continuous functions of $h>0$ and $\|\cdot\|$ is the usual Euclidean norm. Put

$$\sigma(d,h)=\max_{1\le j\le d}\sigma_j(h)$$

and assume that, for some $\alpha>0$, $\sigma(d,h)/h^\alpha$ is *quasi-increasing*, that is, there is a constant $c>0$ such that $\sigma(d,s)/s^\alpha\le c\,\sigma(d,t)/t^\alpha$ for $0<s<t<\infty$.

Let $\{X^d(\mathbf{t})=(X_1(\mathbf{t}),\cdots,X_d(\mathbf{t})),\ \mathbf{t}\in[0,\infty)^N\}$ be a d-dimensional Gaussian process with norm $\|\cdot\|$ and N parameters $t_1,\cdots,t_N\in[0,\infty)$, where $\mathbf{t}=(t_1,\cdots,t_N)$. We call the process $\{X^d(\mathbf{t}),\ \mathbf{t}\in[0,\infty)^N\}$ an (N,d)-*Gaussian random field*. The realizations of random fields $\{X_j(\mathbf{t}),\ \mathbf{t}\in[0,\infty)^N\}$ for $j=1,2,\cdots,d$, are assumed to be different objects. Moreover, the choice of coordinates of the parameter $\mathbf{t}=(t_1,\cdots,t_N)$ is not necessarily limited to length and time but any scale of measurement might be involved.

Now, we introduce some notations to be used in this paper. Let $\mathbf{t}=(t_1,\cdots,t_N)$ and $\mathbf{s}=(s_1,\cdots,s_N)$ be vectors in $[0,\infty)^N$. Denote:

$$\begin{aligned}
&\mathbf{0}=(0,\cdots,0)\ \text{ and }\ \mathbf{1}=(1,\cdots,1)\ \text{ in }\ [0,\infty)^N,\\
&\mathbf{t}\le\mathbf{s}\quad\text{if}\ \ t_i\le s_i\ \ \text{for all integers}\ \ 1\le i\le N,\\
&\mathbf{t}\pm\mathbf{s}=(t_1\pm s_1,\cdots,t_N\pm s_N),\quad \mathbf{ts}=(t_1s_1,\cdots,t_Ns_N),\\
&a\mathbf{t}=(at_1,\cdots,at_N)\quad\text{for } a\in(-\infty,\infty),\\
&\mathbf{a}(T)=(a_1(T),\cdots,a_N(T)),\quad \mathbf{b}(T)=(b_1(T),\cdots,b_N(T)),\\
&\gamma_1(T)=\Big\{2\Big(N\log(\|\mathbf{b}(T)\|/\|\mathbf{a}(T)\|)+\log\log\|\mathbf{b}(T)\|\Big)\Big\}^{1/2},\\
&\gamma_2(T)=\Big\{2\Big(N\log(\|\mathbf{b}(T)\|/\|\mathbf{a}(T)\|)+\log_\theta|\log\|\mathbf{b}(T)\||\Big)\Big\}^{1/2},\\
&\gamma_3(T)=\Big\{2N\log(\|\mathbf{b}(T)\|/\|\mathbf{a}(T)\|)\Big\}^{1/2},
\end{aligned}$$

where $a_i(T)$ and $b_i(T)$, $i=1,\cdots,d$, are positive functions of $T>0$, $\log x=\ln(\max\{x,1\})$ and $1<\theta<e$. Here, $\gamma_2(T)$ will give sharper liminf results than the case of $\gamma_1(T)$ when we state Theorem 1.4 below.

The following results generalize some main theorems on the increments of one dimensional Gaussian processes with one parameter in Choi (1991), Csáki et al.(1991), Csörgő and Révész ((1978), (1981)), Ortega (1984) and Zhang ((1996), (1997)).

The main results are as follows:

Theorem 1.1. *Assume that*

(i) $$\frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|} + \|\mathbf{a}(T)\| \to \infty \quad as \quad T \to \infty.$$

Then we have

$$\limsup_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\gamma_1(T)} \le 1 \qquad \text{a.s.} \tag{1.1}$$

Remark 1.1. We take $\|\cdot\|$ under sup in (1.1) and hereafter, because we have $\{(\mathbf{t},\mathbf{s}) : \mathbf{t} \le \mathbf{b}(T),\, \mathbf{s} \le \mathbf{a}(T))\} \subset \{(\mathbf{t},\mathbf{s}) : \|\mathbf{t}\| \le \|\mathbf{b}(T)\|,\, \|\mathbf{s}\| \le \|\mathbf{a}(T)\|\}$.

The condition (i) implies that $\mathbf{a}(T)$ and $\mathbf{b}(T)$ may be many diverse functions. However, in order to obtain the opposite inequality of (1.1), the conditions on $\mathbf{a}(T)$, $\mathbf{b}(T)$ and $\sigma(d,\cdot)$ are a little bit restricted as in the following Theorem 1.2.

A positive function $\sigma(h)$, $h > 0$, is said to be *regularly varying* with exponent $\alpha > 0$ at $b \ge 0$ if $\lim_{h\to b}\{\sigma(xh)/\sigma(h)\} = x^\alpha$, $x > 0$.

The following theorem is a new version for obtaining the law of iterated logarithm for (N, d)-Gaussian random fields in depending situations.

Theorem 1.2. *For each $i = 1, 2, \cdots, N$, let $a_i(T)$ and $b_i(T)$ be nondecreasing continuous functions such that $\lim_{T\to\infty}\|\mathbf{b}(T)\| = \infty$. Further, assume that $b_i(T)/a_i(T)$ are increasing or $a_i(T) = b_i(T)$ for each $i = 1, 2, \cdots, N$. Suppose that $\sigma(d, h)$ is a regularly varying function with exponent α $(0 < \alpha < 1)$ at ∞ and that there exist positive constants c_1 and c_2 such that, for $h > 0$,*

(ii) $$\left|\frac{d\sigma^2(d,h)}{dh}\right| \le c_1 \frac{\sigma^2(d,h)}{h} \quad and \quad \left|\frac{d^2\sigma^2(d,h)}{dh^2}\right| \le c_2 \frac{\sigma^2(d,h)}{h^2}.$$

Then we have

$$\limsup_{T\to\infty} \frac{\|X^d(\mathbf{b}(T) + \mathbf{a}(T)) - X^d(\mathbf{b}(T))\|}{\sigma(d, \|\mathbf{a}(T)\|)\gamma_1(T)} \ge 1 \qquad \text{a.s.} \tag{1.2}$$

The class of variance functions σ^2 satisfying (ii) contains all concave functions with $0 < \alpha \le 1/2$ (e.g. $\sigma^2(d, h) = \sqrt{h}$) and convex functions with $1/2 < \alpha < 1$. We recall that the correlation function on increments of a stochastic process with stationary increments is nonpositive if and only if its variance function is nearly concave (cf. see (2.10) of this paper, (3.10) and (4.2) in Csáki et al.(1991) and (2.7) in Lin and Qin (1998)), and vice versa.

From Theorems 1.1 and 1.2, we obtain the following lim sup result:

Corollary 1.1. *If conditions of Theorem 1.2 are satisfied, then*

$$\limsup_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_1(T)} = 1 \quad \text{a.s.},$$

$$\limsup_{T\to\infty} \frac{\|X^d(\mathbf{b}(T)+\mathbf{a}(T))-X^d(\mathbf{b}(T))\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_1(T)} = 1 \quad \text{a.s.}$$

If, furthermore, $\|\mathbf{a}(T)\| = \|\mathbf{b}(T)\|$ in Corollary 1.1, then it follows from Theorem 1.1 and Lemma 2.4 of this paper that we have *the law of iterated logarithm* for (N,d)-Gaussian random fields:

Corollary 1.2. (The law of iterated logarithm) *Assume that $b_i(T)$, $i=1,\cdots,N$, are nondecreasing continuous functions such that $\lim_{T\to\infty}\|\mathbf{b}(T)\|=\infty$. Suppose that $\sigma(d,h)$ is a regularly varying function with exponent α $(0<\alpha<1)$ at ∞. Then we have*

$$\limsup_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{b}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{b}(T)\|)\sqrt{2\log\log\|\mathbf{b}(T)\|}} = 1 \quad \text{a.s.},$$

$$\limsup_{T\to\infty} \frac{\|X^d(\mathbf{b}(T))\|}{\sigma(d,\|\mathbf{b}(T)\|)\sqrt{2\log\log\|\mathbf{b}(T)\|}} = 1 \quad \text{a.s.}$$

From now on, we will show that liminf results differ from their corresponding limsup results under the following condition (iii):

Theorem 1.3. *Suppose that, as $T\to\infty$,*

(iii) $\|\mathbf{b}(T)\|\to\infty$ (*or* 0) *and* $\dfrac{\log(\|\mathbf{b}(T)\|/\|\mathbf{a}(T)\|)}{\log_\theta|\log\|\mathbf{b}(T)\||} \to r$ $(0\le r\le\infty)$.

Then we have

$$\liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_2(T)} \le \Big(\frac{Nr}{1+Nr}\Big)^{1/2} \quad \text{a.s.} \tag{1.3}$$

Theorem 1.4. *Assume that $\sigma(d,h)$ is a regularly varying function with exponent α $(0<\alpha<1)$ at 0 or ∞ and that conditions* (ii) *and* (iii) *are satisfied. Then we have*

$$\liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{a}(T))-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_2(T)} \ge \Big(\frac{Nr}{1+Nr}\Big)^{1/2} \quad \text{a.s.} \tag{1.4}$$

The condition (iii) guarantees that the class of vector functions $\mathbf{a}(T)$ and $\mathbf{b}(T)$ contains many various functions (cf. Book and Shore (1978)). Moreover, we assert that the condition (iii) and $\gamma_2(T)$ are essential to show the difference between limsup and liminf results in other stochastic random fields as well as in this paper.

Combining Theorems 1.3 and 1.4 yields the following liminf result, which is distinguished from Corollary 1.1:

Corollary 1.3. *Under the assumptions of Theorem 1.4, we have*

$$\begin{aligned}\liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} & \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_2(T)} \\ &= \liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{a}(T))-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_2(T)} \\ &= \Big(\frac{Nr}{1+Nr}\Big)^{1/2} \quad \text{a.s.}\end{aligned}$$

and, equivalently,

$$\begin{aligned}\liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} & \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_1(T)} \\ &= \liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{a}(T))-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_1(T)} \\ &= \Big(\frac{Nr}{Nr+\log\theta}\Big)^{1/2} \quad \text{a.s.}\end{aligned}$$

Notice that, if $r\to\infty$ in (iii), then Theorem 1.1 and Corollary 1.3 imply that we have the limit result:

$$\begin{aligned}\lim_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} & \frac{\|X^d(\mathbf{t}+\mathbf{a}(T))-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_1(T)} \\ &= \lim_{T\to\infty} \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_1(T)} \\ &= 1 \quad \text{a.s.}\end{aligned} \tag{1.5}$$

under the conditions of Theorem 1.4. For one-parameter Wiener process with $\sigma(1,h) = \sqrt{h}$, the similar results as (1.5) can be found in Csörgő and Révész (1981).

The above results can be applied to develop the limit theories on increments of finite dimensional multiparameter random fields with respect to the following stochastic processes: Ornstein-Uhlenbeck process (e.g. Csáki et al.(1991)), renewal process (Steinebach (1998)), lag sum process (Choi and Hwang (2000)), local-time process (Csörgő et al.(1995)), partial sum process (Szyszkowicz (1993), Steinebach (1983), Deheuvels and Steinebach (1987), Csörgő et al.(1999)), self-normalized partial sum process (Shao (1998), Csörgő et al.(2003)) and etc.

Example 1.1. Let $\{X_j(\mathbf{t}),\ \mathbf{t} \in [0.\infty)^N\}$, $j = 1, 2, \cdots, d$, be N-parameter fractional Brownian motions of orders α_j with $0 < \alpha_j < 1$, that is, let $\{X_j(\mathbf{t}),\ \mathbf{t} \in [0.\infty)^N\}$, $j = 1, 2, \cdots, d$, be Gaussian random fields with $X_j(\mathbf{0}) = 0$ and $\sigma_j(h) = h^{\alpha_j}$, $h > 0$. When $\alpha_j = 1/2$, then $\{X_j(\mathbf{t}),\ \mathbf{t} \in [0.\infty)^N\}$ are standard Wiener random fields. For convenience, put

$$\mathbf{b}(T) = (e^T, \sqrt{2}e^T, \cdots, \sqrt{N}e^T), \quad \mathbf{a}(T) = \mathbf{b}(T)(\log \|\mathbf{b}(T)\|)^{-1}(\log\log T)^{-1},$$

where $e < T < \infty$. Then $\sigma_j(h)$, $\mathbf{a}(T)$ and $\mathbf{b}(T)$ satisfy all conditions of Corollaries 1.1 and 1.3 with

$$\|\mathbf{b}(T)\| = \sqrt{N(N+1)/2}\ e^T =: b_N\, e^T, \quad \gamma_1(T) \approx \{2(N+1)\log T\}^{1/2}$$

for sufficiently large T. Thus we have, by Corollary 1.1,

$$\begin{aligned} &\limsup_{T\to\infty} \sup_{\|\mathbf{t}\| \le b_N e^T} \sup_{\|\mathbf{s}\| \le \|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\delta(T)} \\ &= \limsup_{T\to\infty} \frac{\|X^d(\mathbf{b}(T)+\mathbf{a}(T)) - X^d(\mathbf{b}(T))\|}{\delta(T)} \\ &= {b_N}^{\alpha}\{2(N+1)\}^{1/2} \qquad \text{a.s.,} \end{aligned}$$

where $\delta(T) = \{e^T\, T^{-1}(\log\log T)^{-1}\}^{\alpha}\sqrt{\log T}$, $\alpha = \max_{1\le j\le d}\alpha_j$ and, by Corollary 1.3,

$$\begin{aligned} &\liminf_{T\to\infty} \sup_{\|\mathbf{t}\| \le b_N e^T} \sup_{\|\mathbf{s}\| \le \|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\delta(T)} \\ &= \liminf_{T\to\infty} \sup_{\|\mathbf{t}\| \le b_N e^T} \frac{\|X^d(\mathbf{t}+\mathbf{a}(T)) - X^d(\mathbf{t})\|}{\delta(T)} \\ &= {b_N}^{\alpha}\Big\{\frac{2Nr(N+1)}{Nr+\log\theta}\Big\}^{1/2} \qquad \text{a.s.} \end{aligned}$$

2. Proofs

To prove Theorem 1.1, we need the following lemma (cf. Lin and Choi(2001)):

Lemma 2.1. *For any $\varepsilon > 0$ there exists a positive constant C depending only on ε such that*

$$P\Big\{ \sup_{\|\mathbf{t}\| \le \|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\| \le \|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)} \ge x \Big\} \le C \Big(\frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|} \Big)^N \Phi_d\Big(\frac{2x}{2+\varepsilon} \Big)$$

for all $x > 1$, where $\Phi_d(x) = P\{\|N^d(0,1)\| \ge x\}$ and $N^d(0,1)$ is a d-dimensional standardized normal random vector.

It is well-known that

$$\Phi_d(x) \le c\, x^{d-2} e^{-x^2/2}, \quad x > 1$$

for some $c > 0$ (cf. Lemma 1 in Kôno (1996)).

Proof of Theorem 1.1. Let $\theta = 1 + \varepsilon$ for any given $\varepsilon > 0$. Define

$$\begin{aligned}
&E_k = \{T : \theta^k \le \sigma(d, \|\mathbf{a}(T)\|) \le \theta^{k+1}\}, \quad -\infty < k < \infty, \\
&E_{k,j} = \{T : 2^j \le \frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|} \le 2^{j+1}, T \in E_k\}, \quad 0 \le j < \infty, \\
&\|\mathbf{a}_{T_{k,j}}\| = \sup\{\|\mathbf{a}(T)\| : T \in E_{k,j}\}, \\
&\|\mathbf{b}_{T_{k,j}}\| = \sup\{\|\mathbf{b}(T)\| : T \in E_{k,j}\}.
\end{aligned}$$

By the condition (i), we have

$$\begin{aligned}
&\limsup_{T\to\infty} \sup_{\|\mathbf{t}\| \le \|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\| \le \|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\gamma_1(T)} \\
&\quad \le \limsup_{|k|+l\to\infty} \sup_{j \ge l \ge 0} \sup_{T \in E_{k,j}} \sup_{\|\mathbf{t}\| \le \|\mathbf{b}(T)\|} \sup_{\|\mathbf{s}\| \le \|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\gamma_1(T)} \qquad (2.1) \\
&\quad \le \limsup_{|k|+l\to\infty} \sup_{j \ge l} \sup_{\|\mathbf{t}\| \le \|\mathbf{b}_{T_{k,j}}\|} \sup_{\|\mathbf{s}\| \le \|\mathbf{a}_{T_{k,j}}\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\theta^k\, D(k,j)},
\end{aligned}$$

where $D(k,j)=\big\{2\big(\log 2^{Nj}+\log\log\theta^{|k|+j\log_\theta 2}\big)\big\}^{1/2}$. We will show that

$$\begin{aligned}\limsup_{|k|+l\to\infty}\sup_{j\ge l}\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{T_{k,j}}\|}\sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{T_{k,j}}\|}&\frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\theta^k\,D(k,j)}\\ &\le\theta\limsup_{|k|+l\to\infty}\sup_{j\ge l}\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{T_{k,j}}\|}\sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{T_{k,j}}\|}\frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}_{T_{k,j}}\|)\,D(k,j)}\\ &\le\theta^2\qquad\text{a.s.}\end{aligned}\tag{2.2}$$

By Lemma 2.1, there exists $C_\varepsilon>0$, depending only on $\varepsilon>0$, such that

$$\begin{aligned}&P\Big\{\sup_{j\ge l}\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{T_{k,j}}\|}\sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{T_{k,j}}\|}\frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}_{T_{k,j}}\|)D(k,j)}\ge\theta\Big\}\\ &\le C_\varepsilon\sum_{j\ge l}\Big(\frac{\|\mathbf{b}_{T_{k,j}}\|}{\|\mathbf{a}_{T_{k,j}}\|}\Big)^N\exp\Big(-\frac{2(1+\varepsilon)}{2+\varepsilon}\big(\log 2^{Nj}+\log\log\theta^{|k|+j\log_\theta 2}\big)\Big)\\ &\le C_\varepsilon\sum_{j\ge l}2^{-\varepsilon' Nj}|k\vee 1|^{-1-\varepsilon'}\\ &\le C_\varepsilon\,|k\vee 1|^{-1-\varepsilon'}\,2^{-\varepsilon' Nl}\end{aligned}\tag{2.3}$$

for $|k|+l$ large enough, where $\varepsilon'=\varepsilon/(2+\varepsilon)$ and $k\vee 1=\max\{k,1\}$. Hence we have

$$\sum_{l=0}^{\infty}\sum_{|k|=1}^{\infty}P\Big\{\sup_{j\ge l}\sup_{\|\mathbf{t}\|\le\|\mathbf{b}_{T_{k,j}}\|}\sup_{\|\mathbf{s}\|\le\|\mathbf{a}_{T_{k,j}}\|}\frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}_{T_{k,j}}\|)D(k,j)}\ge\theta\Big\}<\infty,$$

and (2.2) follows from the Borel-Cantelli lemma. Combining (2.2) with (2.1) yields (1.1) by the arbitrariness of θ. □

The following Lemmas 2.2-2.5 are essential for the proof of Thoerem 1.2, and Lemma 2.2 is a well-known version of the second Borel-Cantelli lemma:

Lemma 2.2. *Let $\{A_k, k\ge 1\}$ be a sequence of events. If*

(a) $\displaystyle\sum_{k=1}^{\infty}P(A_k)=\infty$,

(b) $\displaystyle\liminf_{n\to\infty}\sum_{1\le j<k\le n}\frac{P(A_j\cap A_k)-P(A_j)P(A_k)}{\big(\sum_{j=1}^{n}P(A_j)\big)^2}\le 0$,

then $P(A_n,\ \text{i.o.})=1$.

Lemma 2.3. (Berman (1964)) *Let* $\{X_j, j = 1, 2, \cdots, n\}$ *be centered and stationary normal random variables with* $E(X_i X_j) = r_{ij}$ *and* $r_{ii} = 1$. *Let* $I_c^{+1} = [c, \infty)$ *and* $I_c^{-1} = (-\infty, c)$. *Denote by* F_j *the event* $\{X_j \in I_{c_j}^{\varepsilon_j}\}$ *for* $c_j \in (-\infty, \infty)$, $j = 1, 2, \cdots, n$, *where* ε_j *is either* $+1$ *or* -1. *Let* $K \subset \{1, 2, \cdots, n\}$, *then we have the following:*

(i) $P\Big\{\bigcap_{j\in K} F_j\Big\}$ *is an increasing function of* r_{ij} *if* $\varepsilon_i \varepsilon_j = +1$; *otherwise, it is decreasing.*

(ii) *If* $\{K_l, l = 1, 2, \cdots, s\}$ *is a partition of* K, *then*

$$\left|P\Big\{\bigcap_{j\in K} F_j\Big\} - \prod_{l=1}^{s} P\Big\{\bigcap_{j\in K_l} F_j\Big\}\right| \le \sum_{1\le l<m\le s} \sum_{i\in K_l} \sum_{j\in K_m} |r_{ij}| \phi(c_i, c_j; r_{ij}^*),$$

where $\phi(x, y; r)$ *is the standard bivariate normal density with correlation* r, *and* r_{ij}^* *is a number between* 0 *and* r_{ij}.

Lemma 2.4. *Assume that* $b_i(T)$, $i = 1, \cdots, N$, *are nondecreasing continuous functions such that* $\lim_{T\to\infty} \|\mathbf{b}(T)\| = \infty$. *Suppose that* $\sigma(d, h)$ *is a regularly varying function with exponent* α $(0 < \alpha < 1)$ *at* ∞. *Then we have*

$$\limsup_{T\to\infty} \frac{\|X^d(\mathbf{b}(T))\|}{\sigma(d, \|\mathbf{b}(T)\|)\sqrt{2\log\log\|\mathbf{b}(T)\|}} \ge 1 \quad \text{a.s.}$$

Proof. We can find an integer i_0 $(1 \le i_0 \le d)$ such that $\sigma_{i_0}(\|\mathbf{b}(T)\|) = \sigma(d, \|\mathbf{b}(T)\|)$, where i_0 depends on $\|\mathbf{b}(T)\|$. It is clear that

$$\|X^d(\mathbf{b}(T))\|/\sigma(d, \|\mathbf{b}(T)\|) \ge X_{i_0}(\mathbf{b}(T))/\sigma_{i_0}(\|\mathbf{b}(T)\|) =: Y_0(T).$$

For $\theta > 1$, let $\|\mathbf{b}(T_k)\| = \theta^k$, $k = 1, 2, \cdots$. Put $x_k = \sqrt{2(1-\varepsilon)\log\log\theta^k}$ for $0 < \varepsilon < 1$. Setting $A_k = \{Y_0(T_k) > x_k\}$, then

$$P\big(A_k\big) \ge \frac{1}{\sqrt{2\pi}} \frac{\exp\big(-(1-\varepsilon)\log\log\theta^k\big)}{\sqrt{2(1-\varepsilon)\log\log\theta^k}} \ge c\,k^{-(1-\varepsilon/2)}$$

for sufficiently large k, where $c > 0$ is a constant. Thus we have $\sum_{k=1}^{\infty} P\big(A_k\big) = \infty$.

Next, let us show that the condition (b) of Lemma 2.2 is satisfied. For $j < k$, if $i_0(\|\mathbf{b}(T_j)\|) \neq i_0(\|\mathbf{b}(T_k)\|)$, then $E\{X_{i_0}(\mathbf{b}(T_j))X_{i_0}(\mathbf{b}(T_k))\} = 0$; but, if $i_0(\|\mathbf{b}_j\|) = i_0(\|\mathbf{b}_k\|)$, where $\mathbf{b}_k = \mathbf{b}(T_k)$, then

$$
\begin{aligned}
\big|E\big\{X_{i_0}(\mathbf{b}_j)X_{i_0}(\mathbf{b}_k)\big\}\big| &= \frac{1}{2}\big|\sigma_{i_0}^2(\|\mathbf{b}_j\|) + \sigma_{i_0}^2(\|\mathbf{b}_k\|) - \sigma_{i_0}^2(\|\mathbf{b}_k - \mathbf{b}_j\|)\big| \\
&\le \frac{1}{2}\Big(\sigma_{i_0}^2(\|\mathbf{b}_j\|) + \Big(1 - \frac{\|\mathbf{b}_k - \mathbf{b}_j\|^2}{\|\mathbf{b}_k\|^2}\Big)\sigma_{i_0}^2(\|\mathbf{b}_k\|)\Big) \\
&< \frac{1}{2}\Big(\sigma_{i_0}^2(\|\mathbf{b}_j\|) + \frac{2\|\mathbf{b}_j\|}{\|\mathbf{b}_k\|}\sigma_{i_0}^2(\|\mathbf{b}_k\|)\Big)
\end{aligned}
$$

and further

$$
\begin{aligned}
|r_{jk}| := |E\{Y_0(T_j)Y_0(T_k)\}| &< \frac{\sigma_{i_0}(\|\mathbf{b}_j\|)}{2\sigma_{i_0}(\|\mathbf{b}_k\|)} + \frac{\|\mathbf{b}_j\|\sigma_{i_0}(\|\mathbf{b}_k\|)}{\|\mathbf{b}_k\|\sigma_{i_0}(\|\mathbf{b}_j\|)} \\
&= \frac{1}{2}\Big(\frac{\|\mathbf{b}_j\|}{\|\mathbf{b}_k\|}\Big)^{\alpha_{i_0}}\frac{L_{i_0}(\|\mathbf{b}_j\|)}{L_{i_0}(\|\mathbf{b}_k\|)} + \Big(\frac{\|\mathbf{b}_j\|}{\|\mathbf{b}_k\|}\Big)^{1-\alpha_{i_0}}\frac{L_{i_0}(\|\mathbf{b}_k\|)}{L_{i_0}(\|\mathbf{b}_j\|)} \\
&\le \frac{1}{2}\theta^{-\alpha'(k-j)} + \theta^{-(1-\alpha'')(k-j)} =: \eta_{jk},
\end{aligned}
$$

where $L_{i_0}(\cdot)$ is a slowly varying function and $0 < \alpha' < \alpha_{i_0} < \alpha'' < 1$. Let $\alpha_0 = \min\{\alpha', 1-\alpha''\}$ and $\xi_l = \big[\frac{2}{\alpha_0}\log_\theta l\big]$, where we take l $(j < l \le k)$ such that ${x_l}^2 \ge x_j x_k$. It follows from Lemma 2.3 that, for sufficiently large n,

$$
\begin{aligned}
&\sum_{1\le j<k\le n}\Big\{P(A_j \cap A_k) - P(A_j)P(A_k)\Big\} \\
&\quad\le \sum_{j=1}^{n-1}\sum_{k=j+1}^{n}\frac{|r_{jk}|}{2\pi(1-{r_{jk}^*}^2)^{1/2}}\exp\Big\{-\frac{x_j^2 + x_k^2 - 2x_jx_k|r_{jk}^*|}{2(1-|r_{jk}^*|^2)}\Big\} \\
&\quad\le \sum_{j=1}^{n-1}\sum_{k=j+1}^{j+\xi_l}\frac{|r_{jk}|}{2\pi(1-{r_{jk}^*}^2)^{1/2}}e^{-x_j^2/2}\exp\Big\{-\Big(\frac{1-|r_{jk}^*|}{1+|r_{jk}^*|}\Big)\frac{x_l^2}{2}\Big\} \\
&\quad\quad + \sum_{j=1}^{n-1}\sum_{k=j+\xi_l+1}^{n}\frac{|r_{jk}|}{2\pi(1-{r_{jk}^*}^2)^{1/2}}e^{-x_k^2/2}e^{-x_j^2/2}\exp\Big\{\frac{1}{1-|r_{jk}^*|^2}|r_{jk}^*|x_l^2\Big\} \\
&\quad =: S_1 + S_2.
\end{aligned}
$$

Consider the first sum S_1. If r $(0 < r < 1)$ is the maximum of $|r_{jk}|$ for $1 \le j < k \le n$, then we have

$$
\begin{aligned}
S_1 &\le \sum_{j=1}^{n-1}\sum_{k=j+1}^{j+\xi_l}\frac{r}{2\pi\sqrt{1-r^2}}e^{-x_j^2/2}\exp\Big\{-\Big(\frac{1-r}{1+r}\Big)\frac{x_l^2}{2}\Big\}\\
&\le c\sum_{j=1}^{n-1}\xi_l\,e^{-x_j^2/2}\exp\Big\{-R\frac{2(1-\varepsilon)\log\log\theta^l}{2}\Big\}
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
&\le c\sum_{j=1}^{n-1}e^{-x_j^2/2}\,(\log_\theta l)\,l^{-R(1-\varepsilon)}\le c\sum_{j=1}^{n-1}e^{-x_j^2/2}\frac{1}{\sqrt{2\pi}\,x_j^2}\,l^{-R(1-\varepsilon)/2}\\
&\le c\sum_{j=1}^{n}P(A_j)\,l^{-R(1-\varepsilon)/2},
\end{aligned}
$$

where $R:=(1-r)/(1+r)$ and $c>0$ is a relevant constant. Next, consider the second sum S_2. For $k-j>\xi_l$, since

$$
\begin{aligned}
|r_{jk}|x_l^2 \le \eta_{jk}x_l^2 &\le \Big(1-\frac{\varepsilon}{2}\Big)\Big(l^{-2\alpha'/\alpha_0}+2\,l^{-2(1-\alpha'')/\alpha_0}\Big)\log(l\,\log\theta)\\
&\le 3\Big(1-\frac{\varepsilon}{2}\Big)\,l^{-2}\log(l\,\log\theta),
\end{aligned}
$$

we have

$$
S_2\le c\,l^{-1}\sum_{j=1}^{n-1}\sum_{k=1}^{n}\frac{1}{2\pi {x_j}^2{x_k}^2}\,e^{-({x_j}^2+{x_k}^2)/2}\le c\,l^{-1}\Big(\sum_{j=1}^{n}P(A_j)\Big)^2. \tag{2.5}
$$

From (2.4) and (2.5), we obtain

$$
\begin{aligned}
&\sum_{1\le j<k\le n}\Big\{P\big(A_j\cap A_k\big)-P\big(A_j\big)P\big(A_k\big)\Big\}\\
&\qquad\le c\,l^{-R(1-\varepsilon)/2}\Big(\sum_{j=1}^{n}P\big(A_j\big)+\Big(\sum_{j=1}^{n}P(A_j)\Big)^2\Big),
\end{aligned}
$$

and the condition (b) of Lemma 2.2 is satisfied whenever $l\to\infty$. $\Box$

Lemma 2.5. *Assume that the condition* (ii) *of Theorem 1.2 is satisfied. For* $i = 0, 1, 2, 3$*, let* $\mathbf{a}^{(i)} = (a_1^{(i)}, \cdots, a_N^{(i)})$ *be positive* N*-dimensional vectors such that* $a_j^{(3)} - a_j^{(2)} > a_j^{(2)} - a_j^{(1)} > 0$ *for each* $j = 1, 2, \cdots, N$*. Then there exists a positive constant* C *such that*

$$\begin{aligned}
&\int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|} d\sigma^2(d,x) - \int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} d\sigma^2(d,x) \\
&\qquad \le C\, \frac{\sigma^2\big(d, \|\mathbf{a}^{(0)} + \mathbf{a}^{(3)}\|\big)\|\mathbf{a}^{(3)} - \mathbf{a}^{(2)}\|\|\mathbf{a}^{(2)} - \mathbf{a}^{(1)}\|}{\|\mathbf{a}^{(0)} + \mathbf{a}^{(1)}\|^2}\,.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|} d\sigma^2(d,x) - \int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} d\sigma^2(d,x) \\
&= \int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|+\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} \Big(\frac{d\sigma^2(d, x + \|\mathbf{a}^{(0)} + \mathbf{a}^{(2)}\| - \|\mathbf{a}^{(0)} + \mathbf{a}^{(1)}\|)}{dx} \\
&\qquad - \frac{d\sigma^2(d,x)}{dx}\Big)dx + \int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|+\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} \Big(\frac{d\sigma^2(d,x)}{dx}\Big)dx \\
&\le \int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|+\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} \int_x^{x+\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|} \Big|\frac{d^2\sigma^2(d,y)}{dy^2}\Big| dy dx \\
&\qquad + \int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|+\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} \Big|\frac{d\sigma^2(d,x)}{dx}\Big| dx \\
&=: I + J, \quad \text{say.}
\end{aligned}$$

Thus,

$$
\begin{aligned}
I &\le \int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|+\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} \int_{x}^{x+\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|} \Big(c_2 \frac{\sigma^2(d,y)}{y^2}\Big) dy dx \\
&\le c_2 \int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|+\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} \frac{\sigma^2(d, x+\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|)}{x^2} \\
&\qquad\qquad \times (\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|) dx \\
&\le c_2 \frac{\sigma^2(d, \|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|)}{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|^2} (\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|) \\
&\qquad\qquad \times (\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|) \\
&\le c_2 \frac{\sigma^2(d, \|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|)}{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|^2} \|\mathbf{a}^{(3)}-\mathbf{a}^{(2)}\| \, \|\mathbf{a}^{(2)}-\mathbf{a}^{(1)}\|
\end{aligned}
$$

and

$$
\begin{aligned}
J &\le c_1 \int_{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|}^{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|+\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} \Big(\frac{\sigma^2(d,x)}{x}\Big) dx \\
&\le c_1 \frac{\sigma^2(d, \|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|)}{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} \\
&\qquad \times \big(\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|-(\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|-\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|)\big) \\
&\le c_1 \frac{\sigma^2(d, \|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|)}{\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} \\
&\qquad \times \frac{\|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|^2-\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|^2-(\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|^2-\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|^2)}{2\|\mathbf{a}^{(0)}+\mathbf{a}^{(2)}\|} \\
&\le c \, \frac{\sigma^2(d, \|\mathbf{a}^{(0)}+\mathbf{a}^{(3)}\|)}{\|\mathbf{a}^{(0)}+\mathbf{a}^{(1)}\|^2} \|\mathbf{a}^{(3)}-\mathbf{a}^{(2)}\| \, \|\mathbf{a}^{(2)}-\mathbf{a}^{(1)}\|,
\end{aligned}
$$

where $c > 0$ is a constant. $\square$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. If $a_i(T) = b_i(T)$ for each $i = 1, 2, \cdots, N$, then (1.2) is immediate from Lemma 2.4. In what follows, let $b_i(T)/a_i(T)$ be increasing for each $i = 1, 2, \cdots, N$. Clearly, there exists an integer j_0 $(1 \le j_0 \le d)$ such that $\sigma_{j_0}(\|\mathbf{a}(T)\|) = \sigma(d, \|\mathbf{a}(T)\|)$. Thus

$$\begin{aligned}\limsup_{T\to\infty} & \frac{\|X^d(\mathbf{b}(T)) - X^d(\mathbf{b}(T) - \mathbf{a}(T))\|}{\sigma(d, \|\mathbf{a}(T)\|)\gamma_1(T)} \\ & \ge \limsup_{T\to\infty} \frac{X_{j_0}(\mathbf{b}(T)) - X_{j_0}(\mathbf{b}(T) - \mathbf{a}(T))}{\sigma_{j_0}(\|\mathbf{a}(T)\|)\gamma_1(T)}.\end{aligned} \tag{2.6}$$

Without loss of generality, let $T_1 = 1$. Noting that $\|\mathbf{b}(T) - \mathbf{a}(T)\|$ is increasing, we can define an increasing sequence $\{T_i\}_{i=2}^{\infty}$ such that

$$\left\|\mathbf{b}(T_i) - \mathbf{a}(T_i) - \sum_{j=1}^{i-1} \mathbf{a}(T_j)\right\| = \|\mathbf{b}(T_1)\| \tag{2.7}$$

if T_j $(j = 1, 2, \cdots, i-1)$ have been defined, by induction. Put $\mathbf{a}_i = \mathbf{a}(T_i)$ and $\mathbf{b}_i = \mathbf{b}(T_i)$, $i \ge 1$, and let

$$Z_i := \frac{X_{j_0}(\mathbf{b}_i) - X_{j_0}(\mathbf{b}_i - \mathbf{a}_i)}{\sigma_{j_0}(\|\mathbf{a}_i\|)}.$$

The proof of (1.2) is completed if we show that

$$\limsup_{i\to\infty} \frac{Z_i}{\gamma_1(T_i)} \ge 1 \qquad \text{a.s.} \tag{2.8}$$

For any given $0 < \varepsilon < 1$, let $B_i = \{Z_i > (1-\varepsilon)\gamma_1(T_i)\}$, $i \ge 1$. First we will show that $\sum_{i=1}^{\infty} P(B_i) = \infty$. Put $x_i = (1-\varepsilon)\gamma_1(T_i)$. Then, for sufficiently large i, we have

$$\begin{aligned}P(B_i) &\ge \frac{1}{\sqrt{2\pi}}\Big(\frac{1}{x_i} - \frac{1}{x_i^3}\Big)\exp\Big(-\frac{1}{2}x_i^2\Big) \\ &\ge \exp\Big(-\frac{1}{2-\varepsilon}x_i^2\Big) \ge \Big(\frac{\|\mathbf{a}_i\|^N}{\|\mathbf{b}_i\|^N \log\|\mathbf{b}_i\|}\Big)^{1-\varepsilon}\end{aligned}$$

and

$$\sum_{i=i_0}^{m} P(B_i) \ge \frac{1}{(\log\|\mathbf{b}_m\|)^{1-\varepsilon}} \sum_{i=i_0}^{m} \frac{\|\mathbf{a}_i\|^N}{\|\mathbf{b}_i\|^N}$$

for some i_0 with $i \geq i_0$. Further, we have

$$\begin{aligned}
\log \|\mathbf{b}_m\| &\leq c \sum_{i=i_0}^{m} \log \frac{\|\mathbf{b}_i\|}{\|\mathbf{b}_{i-1}\|} \\
&\leq -c \sum_{i=i_0}^{m} \log \frac{\|\sum_{j=1}^{i-1} \mathbf{a}_j\| - \|\mathbf{b}_1\|}{\|\sum_{j=1}^{i} \mathbf{a}_j\| + \|\mathbf{b}_1\|} \\
&\leq -c \sum_{i=i_0}^{m} \log \Big(1 - \frac{\|\mathbf{a}_i\| + 2\|\mathbf{b}_1\|}{\|\sum_{j=1}^{i} \mathbf{a}_j\| + \|\mathbf{b}_1\|}\Big).
\end{aligned} \tag{2.9}$$

where $c > 1$ is a constant. Since $\|\mathbf{a}_i\|$ is nondecreasing and $\lim_{i\to\infty} \|\mathbf{b}_i\| = \infty$, we can find a constant $c_0 > 1$ such that

$$1 + 2\frac{\|\mathbf{b}_1\|}{\|\mathbf{a}_i\|} \leq c_0 < \frac{\|\mathbf{b}_i\|}{\|\mathbf{a}_i\|}$$

for $i \geq i_0$. It follows from (2.9) that there exists a constant $K > 0$ such that

$$N \log \|\mathbf{b}_m\| \leq -cN \sum_{i=i_0}^{m} \log\Big(1 - \frac{c_0\|\mathbf{a}_i\|}{\|\mathbf{b}_i\|}\Big) \leq K \sum_{i=i_0}^{m} \Big(\frac{c_0\|\mathbf{a}_i\|}{\|\mathbf{b}_i\|}\Big)^N.$$

Therefore, we have

$$\sum_{i=1}^{m} P(B_i) \geq \frac{N}{Kc_0^N} (\log \|\mathbf{b}_m\|)^{\varepsilon} \to \infty \qquad \text{as} \quad m \to \infty.$$

Now, (2.8) is proved if we show that condition (b) of Lemma 2.2 is satisfied. From the elementary relation $ab = (a^2 + b^2 - (a-b)^2)/2$, we have, for all i and j with $i < j$,

$$\begin{aligned}
E(Z_i Z_j) &= \frac{1}{\sigma_{j_0}(\|\mathbf{a}_i\|)\sigma_{j_0}(\|\mathbf{a}_j\|)} E\big\{(X_{j_0}(\mathbf{b}_i) - X_{j_0}(\mathbf{b}_i - \mathbf{a}_i)) \\
&\qquad \times (X_{j_0}(\mathbf{b}_j) - X_{j_0}(\mathbf{b}_j - \mathbf{a}_j))\big\} \\
&= -\frac{1}{2\sigma_{j_0}(\|\mathbf{a}_i\|)\sigma_{j_0}(\|\mathbf{a}_j\|)} \big\{\sigma_{j_0}^2(\|\mathbf{b}_j - \mathbf{b}_i\|) - \sigma_{j_0}^2(\|\mathbf{b}_j - \mathbf{b}_i - \mathbf{a}_j\|) \\
&\qquad - \sigma_{j_0}^2(\|\mathbf{b}_j - \mathbf{b}_i + \mathbf{a}_i\|) + \sigma_{j_0}^2(\|\mathbf{b}_j - \mathbf{b}_i - \mathbf{a}_j + \mathbf{a}_i\|)\big\} \\
&= \frac{1}{2\sigma_{j_0}(\|\mathbf{a}_i\|)\sigma_{j_0}(\|\mathbf{a}_j\|)} \big\{\sigma_{j_0}^2(\|\mathbf{b}_j - \mathbf{b}_i + \mathbf{a}_i\|) - \sigma_{j_0}^2(\|\mathbf{b}_j - \mathbf{b}_i\|) \\
&\qquad - (\sigma_{j_0}^2(\|\mathbf{b}_j - \mathbf{b}_i - \mathbf{a}_j + \mathbf{a}_i\|) - \sigma_{j_0}^2(\|\mathbf{b}_j - \mathbf{b}_i - \mathbf{a}_j\|))\big\}.
\end{aligned} \tag{2.10}$$

If the right hand side of (2.10) is less than or equal to zero, that is, $\sigma_{j_0}^2$ is a nearly convex function with $0<\alpha\le 1/2$, then $P(B_i\cap B_j)\le P(B_i)P(B_j)$ by (i) of Lemma 2.3, and hence (b) of Lemma 2.2 holds true. On the contrary, if the right hand side of (2.10) is larger than zero, then

$$\begin{aligned} E(Z_iZ_j) \le & \frac{1}{\sigma_{j_0}(\|\mathbf{a}_i\|)\sigma_{j_0}(\|\mathbf{a}_j\|)} \\ & \times \Big\{ \int_{\|\mathbf{b}_j-\mathbf{b}_i-\mathbf{a}_j+\mathbf{a}_i\|}^{\|\mathbf{b}_j-\mathbf{b}_i+\mathbf{a}_i\|} d\sigma_{j_0}^2(x) - \int_{\|\mathbf{b}_j-\mathbf{b}_i-\mathbf{a}_j\|}^{\|\mathbf{b}_j-\mathbf{b}_i-\mathbf{a}_j+\mathbf{a}_i\|} d\sigma_{j_0}^2(x)\Big\}. \end{aligned} \tag{2.11}$$

Applying Lemma 2.5 with $\mathbf{a}^{(0)}=\mathbf{b}_j-\mathbf{b}_i-\mathbf{a}_j$, $\mathbf{a}^{(1)}=\mathbf{0}$, $\mathbf{a}^{(2)}=\mathbf{a}_i$ and $\mathbf{a}^{(3)}=\mathbf{a}_j+\mathbf{a}_i$, the right-hand sideof (2.11) is less than or equal to

$$\frac{C\,\sigma_{j_0}^2(\|\mathbf{b}_j-\mathbf{b}_i+\mathbf{a}_i\|)\|\mathbf{a}_i\|\,\|\mathbf{a}_j\|}{\sigma_{j_0}(\|\mathbf{a}_i\|)\sigma_{j_0}(\|\mathbf{a}_j\|)\|\mathbf{b}_j-\mathbf{b}_i-\mathbf{a}_j\|^2}. \tag{2.12}$$

By the definition of $\{T_i\}_{i=2}^{\infty}$, we have

$$\|\mathbf{b}_j-\mathbf{b}_i+\mathbf{a}_i\| = \Big\|\mathbf{b}_j-\sum_{l=1}^{j}\mathbf{a}_l-\Big(\mathbf{b}_i-\sum_{l=1}^{i}\mathbf{a}_l\Big)+\sum_{l=i}^{j}\mathbf{a}_l\Big\| = \Big\|\sum_{l=i}^{j}\mathbf{a}_l\Big\| \tag{2.13}$$

and

$$\|\mathbf{b}_j-\mathbf{b}_i-\mathbf{a}_j\| = \Big\|\sum_{l=i+1}^{j-1}\mathbf{a}_l\Big\| \ge \rho'\,\Big\|\sum_{l=i}^{j}\mathbf{a}_l\Big\|, \quad j\ge i+2 \tag{2.14}$$

for some $0<\rho'<1$. Noting that $\|\mathbf{b}_i\|/\|\mathbf{a}_i\|$ is increasing, then it follows that

$$\frac{\|\mathbf{a}_i\|}{\|\mathbf{a}_j\|} \ge \frac{\|\mathbf{b}_i\|}{\|\mathbf{b}_j\|} \ge 1-\frac{\|\mathbf{b}_j-\mathbf{b}_i\|}{\|\mathbf{b}_j\|} = 1-\frac{\|\sum_{l=i+1}^{j}\mathbf{a}_l\|}{\|\mathbf{b}_j\|} \ge 1-\rho \tag{2.15}$$

for some $0<\rho<1$. From (2.11)-(2.15) and the property of slowly varying function $L(\cdot)$, we have

$$\begin{aligned} E(Z_iZ_j) &\le \frac{c\,\sigma_{j_0}^2(\|\sum_{l=i}^{j}\mathbf{a}_l\|)\|\mathbf{a}_i\|\|\mathbf{a}_j\|}{\sigma_{j_0}(\|\mathbf{a}_i\|)\sigma_{j_0}(\|\mathbf{a}_j\|)\|\sum_{l=i}^{j}\mathbf{a}_l\|^2} \\ &\le c\Big\|\sum_{l=i}^{j}\mathbf{a}_l\Big\|^{2\alpha-2}(\|\mathbf{a}_i\|\|\mathbf{a}_j\|)^{1-\alpha}\,L\Big(\Big\|\sum_{l=i}^{j}\mathbf{a}_l\Big\|\Big)^2(L(\|\mathbf{a}_i\|)L(\|\mathbf{a}_j\|))^{-1} \\ &\le c\,\frac{(j-i)^{2\alpha-2}\|\mathbf{a}_j\|^{1-\alpha}L(\|(j-i)\mathbf{a}_i\|)^2}{L(\|\mathbf{a}_i\|)L(\|\mathbf{a}_j\|)\|\mathbf{a}_i\|^{1-\alpha}} \\ &\le c\,(j-i)^{\alpha-1}, \quad j\ge i+2, \end{aligned}$$

where $c > 0$ is a relevant constant. The remainder of the proof is exactly the same as the corresponding proof of Theorem 2 in Ortega (1984). The details are omitted. $\square$

The following lemma can be found in Lin and Lu (1992), which is used to prove Theorem 1.3:

Lemma 2.6. *Let $\{\xi, \xi_n : n \geq 1\}$ be a sequence of random variables. If $P\{\xi_n \geq \xi\} \to 0$ as $n \to \infty$, then there is a subsequence $\{\xi_{n_k}\}$ such that*

$$\limsup_{k\to\infty} \xi_{n_k} \leq \xi \qquad \text{a.s.}$$

Proof of Theorem 1.3. First, consider the case $0 < r \leq \infty$. From the condition (iii), there exists a positive number γ such that

$$\frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|} \geq |\log \|\mathbf{b}(T)\||^{\gamma/\log\theta},$$

provided T is large enough. Thus, it follows from (iii) and Lemma 2.1 that, for any $\varepsilon > 0$,

$$\begin{aligned} &P\Big\{\sup_{\|\mathbf{t}\|\leq\|\mathbf{b}(T)\|}\ \sup_{\|\mathbf{s}\|\leq\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\gamma_3(T)} > \sqrt{1+2\varepsilon}\Big\} \\ &\quad\leq c\Big(\frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|}\Big)^N \exp\Big(-\frac{4(1+2\varepsilon)}{(2+\varepsilon)^2}\log\Big(\frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|}\Big)^N\Big) \\ &\quad\leq c\,|\log\|\mathbf{b}(T)\||^{-N\gamma\varepsilon/((2+\varepsilon)^2\log\theta)} \to 0 \qquad \text{as} \quad T\to\infty, \end{aligned} \tag{2.16}$$

and Lemma 2.6 implies

$$\liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\leq\|\mathbf{b}(T)\|}\ \sup_{\|\mathbf{s}\|\leq\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\gamma_3(T)} \leq 1 \qquad \text{a.s.}$$

Hence, by (iii), we get

$$\begin{aligned} &\liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\leq\|\mathbf{b}(T)\|}\ \sup_{\|\mathbf{s}\|\leq\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\gamma_2(T)} \\ &\quad= \liminf_{T\to\infty} \sup_{\|\mathbf{t}\|\leq\|\mathbf{b}(T)\|}\ \sup_{\|\mathbf{s}\|\leq\|\mathbf{a}(T)\|} \frac{\|X^d(\mathbf{t}+\mathbf{s}) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\gamma_3(T)}\ \frac{\gamma_3(T)}{\gamma_2(T)} \\ &\quad\leq \Big(\frac{Nr}{1+Nr}\Big)^{1/2} \qquad \text{a.s.} \end{aligned} \tag{2.17}$$

Next, consider the case $r = 0$. It follows from (iii) that, for any $\varepsilon > 0$,

$$\frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|} < |\log\|\mathbf{b}(T)\||^{\varepsilon/((2+\varepsilon)N\log\theta)}$$

for T large enough. Similarly to (2.16), we have

$$\begin{aligned}
&P\Big\{\sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|}\ \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|}\frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_2(T)} > \sqrt{\varepsilon}\Big\}\\
&\quad\le c\Big(\frac{\|\mathbf{b}(T)\|}{\|\mathbf{a}(T)\|}\Big)^N\exp\Big(-\frac{1}{2}\Big(\frac{2}{2+\varepsilon}\Big)^2\varepsilon\,\gamma_2^2(T)\Big)\\
&\quad\le c\,|\log\|\mathbf{b}(T)\||^{-\varepsilon/((4+2\varepsilon)\log\theta)}\to 0 \qquad \text{as} \quad T\to\infty,
\end{aligned}$$

which implies that

$$\liminf_{T\to\infty}\ \sup_{\|\mathbf{t}\|\le\|\mathbf{b}(T)\|}\ \sup_{\|\mathbf{s}\|\le\|\mathbf{a}(T)\|}\frac{\|X^d(\mathbf{t}+\mathbf{s})-X^d(\mathbf{t})\|}{\sigma(d,\|\mathbf{a}(T)\|)\gamma_2(T)}\le 0 \quad \text{a.s.} \tag{2.18}$$

Combining (2.17) with (2.18) completes the proof of Theorem 1.3. □

To prove Theorem 1.4, we need the following Lemmas 2.7-2.10. The proof of Lemma 2.7 is similar to that of Lemma 2.5.

Lemma 2.7. *Assume that the condition* (ii) *of Theorem 1.2 is satisfied. Let* $\mathbf{a} > \mathbf{0}$ *and* $\mathbf{b} > \mathbf{1}$ *be* N*-dimensional vectors. Then there exists a positive constant* C *such that*

$$\Big|\int_{\|\mathbf{a}\|\,\|\mathbf{b}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}+\mathbf{1}\|} d\sigma^2(d,x) - \int_{\|\mathbf{a}\|\,\|\mathbf{b}-\mathbf{1}\|}^{\|\mathbf{a}\|\,\|\mathbf{b}\|} d\sigma^2(d,x)\Big| \le C\,\frac{\sigma^2(d,\|\mathbf{a}\|\|\mathbf{b}+\mathbf{1}\|)}{\|\mathbf{b}-\mathbf{1}\|^2}.$$

Lemma 2.8. (Slepian (1962)) *Suppose that* $\{V_i,\, i=1,\cdots,n\}$ *and* $\{W_i,\, i=1,\cdots,n\}$ *are jointly standardized normal random variable with* $Cov(V_i,V_j)\le Cov(W_i,W_j)$, $i\ne j$. *Then, for any real* $u_i\,(i=1,\cdots,n)$, *we have*

$$P\{V_i\le u_i,\ i=1,\cdots,n\}\le P\{W_i\le u_i,\ i=1,\cdots,n\},$$

$$P\{V_i\ge u_i,\ i=1,\cdots,n\}\le P\{W_i\ge u_i,\ i=1,\cdots,n\}.$$

Lemma 2.9. (Leadbetter et al.(1983)), Li and Shao (2002)) *Let* $\mathbb{N} = (n_1, \cdots, n_N)$ *be a* N*-dimensional vector, where* $n_1, \cdots, n_N = 1, 2, \cdots, L$. *Suppose that* $\{Y(\mathbb{N})\}$ *is a sequence of* N*-parameter standard normal random variables with* $\Lambda(\mathbb{N}, \mathbb{N}') := \operatorname{Cov}(Y(\mathbb{N}), Y(\mathbb{N}'))$ *for* $\mathbb{N} \neq \mathbb{N}'$ *such that*

$$\delta := \max_{\mathbb{N} \neq \mathbb{N}'} |\Lambda(\mathbb{N}, \mathbb{N}')| < 1.$$

Let $\{l_{\mathbb{N}} = (l_{n_1}, \cdots, l_{n_N})\}$ *be a subsequence of* $\{\mathbb{N}\}$. *Denote* $\mathbf{m} = (m_1, \cdots, m_N)$ *with* $m_i \le L$, $1 \le i \le N$. *Then, for any real number* u, *we have*

$$\begin{aligned} &P\Big\{ \max_{\mathbf{1} \le \mathbb{N} \le \mathbf{m}} Y(l_{\mathbb{N}}) \le u \Big\} \\ &\quad \le \big\{\Phi(u)\big\}^{\prod_{i=1}^N m_i} + \sum_{\substack{\mathbb{N} \neq \mathbb{N}' \\ \mathbf{1} \le \mathbb{N}, \mathbb{N}' \le \mathbf{m}}} c\, |\lambda(\mathbb{N}, \mathbb{N}')| \exp\bigg(- \frac{u^2}{1 + |\lambda(\mathbb{N}, \mathbb{N}')|} \bigg) \end{aligned} \tag{2.19}$$

for some $c > 0$, *where* $\lambda(\mathbb{N}, \mathbb{N}') = \Lambda(l_{\mathbb{N}}, l_{\mathbb{N}'})$ *and* $\Phi(u) = \int_{-\infty}^u \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$.

Estimating an upper bound for the second term of the right hand side of (2.19), we obtain the following lemma, whose proof is similar to that of Lemma 7 in Choi and Kôno (1999).

Lemma 2.10. *Let* $Y(\mathbb{N})$, δ *and* $\lambda(\mathbb{N}, \mathbb{N}')$ *be as in Lemma 2.9. Further, assume that the inequality*

$$|\lambda(\mathbb{N}, \mathbb{N}')| < \|\mathbb{N} - \mathbb{N}'\|^{-\nu}$$

holds for some $\nu > 0$. *Set* $u = \big\{(2 - \eta) \log\big(\prod_{i=1}^N m_i\big)\big\}^{1/2}$, *where* $0 < \eta < (1 - \delta)\nu/(1 + \nu + \delta)$. *Then we have*

$$\sum := \sum_{\substack{\mathbb{N} \neq \mathbb{N}' \\ \mathbf{1} \le \mathbb{N}, \mathbb{N}' \le \mathbf{m}}} |\lambda(\mathbb{N}, \mathbb{N}')| \exp\bigg(- \frac{u^2}{1 + |\lambda(\mathbb{N}, \mathbb{N}')|} \bigg) \le c \bigg(\prod_{i=1}^N m_i \bigg)^{-\delta_0},$$

where $\delta_0 = \{\nu(1 - \delta) - \eta(1 + \delta + \nu)\}/\{(1 + \nu)(1 + \delta)\} > 0$ *and* c *is a positive constant independent of* $\mathbb{N}$ *and* u.

Proof of Theorem 1.4. The inequality (1.4) is obvious when $r = 0$. In what follows, we assume that $0 < r \le \infty$. For $1 < \theta < e$, define

$$B_{k,\mathbf{j}} = \{T : \theta^{k-1} \le \|\mathbf{b}(T)\| \le \theta^k, \ \ \theta^{j_i - 1} \le a_i(T) \le \theta^{j_i}, \ 1 \le i \le N\},$$

where k and j_i are integers. Denote $\mathbf{j} = (j_1, \cdots, j_N)$, $\boldsymbol{\theta}^{a\mathbf{j}} = (\theta^{aj_1}, \cdots, \theta^{aj_N})$ for $-\infty < a < \infty$ and $\underline{j} = \frac{1}{N}\sum_{i=1}^N j_i$. In the sequel, we always consider k and $\mathbf{j}$ such that $B_{k,\mathbf{j}} \neq \emptyset$. Note that $\|\mathbf{a}(T)\| \geq \theta^{\underline{j}-1}$ for $T \in B_{k,\mathbf{j}}$. By the condition (iii), there exists $\gamma > 0$ such that

$$\underline{j} \leq k + 1 - \gamma(\log\log\theta^{|k|})/(\log\theta)^2 =: K$$

for k sufficently large. Noting that

$$\lim_{T\to\infty}\frac{\gamma_2(T)}{\gamma_3(T)} = \Big(\frac{Nr+1}{Nr}\Big)^{1/2},$$

the inequality (1.4) is proved if we show that

$$\liminf_{T\to\infty}\sup_{\|\mathbf{t}\|\leq\|\mathbf{b}(T)\|}\frac{\|X^d(\mathbf{t}+\mathbf{a}(T)) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\gamma_3(T)} \geq 1 \qquad \text{a.s.} \tag{2.20}$$

We can write

$$\begin{aligned}
&\liminf_{T\to\infty}\sup_{\|\mathbf{t}\|\leq\|\mathbf{b}(T)\|}\frac{\|X^d(\mathbf{t}+\mathbf{a}(T)) - X^d(\mathbf{t})\|}{\sigma(d, \|\mathbf{a}(T)\|)\,\gamma_3(T)} \\
&\quad\geq \liminf_{|k|\to\infty}\inf_{\underline{j}\leq K}\sup_{\|\mathbf{t}\|\leq\theta^{k-1}}\frac{\|X^d(\mathbf{t}+\boldsymbol{\theta}^{\mathbf{j}}) - X^d(\mathbf{t})\|}{\sigma(d, \|\boldsymbol{\theta}^{\mathbf{j}}\|)\sqrt{2N\log\theta^{k-\underline{j}+1}}} \\
&\quad\quad - \limsup_{|k|\to\infty}\sup_{\underline{j}\leq K}\sup_{\|\mathbf{t}\|\leq\theta^{k}}\sup_{\boldsymbol{\theta}^{\mathbf{j}-\mathbf{1}}\leq\mathbf{s}\leq\boldsymbol{\theta}^{\mathbf{j}}} \\
&\quad\quad\quad \frac{\|X^d(\mathbf{t}+\boldsymbol{\theta}^{\mathbf{j}}) - X^d(\mathbf{t}+\mathbf{s})\|}{\sigma(d, \|\boldsymbol{\theta}^{\mathbf{j}} - \boldsymbol{\theta}^{\mathbf{j}-\mathbf{1}}\|)\sqrt{2N\log\theta^{k-\underline{j}+1}}}\,\frac{\sigma(d, \|\boldsymbol{\theta}^{\mathbf{j}} - \boldsymbol{\theta}^{\mathbf{j}-\mathbf{1}}\|)}{\sigma(d, \|\boldsymbol{\theta}^{\mathbf{j}-\mathbf{1}}\|)} \\
&=: J_1 - J_2.
\end{aligned} \tag{2.21}$$

First, we claim that

$$J_1 \geq 1 \qquad \text{a.s.} \tag{2.22}$$

By the definition of $\sigma(d, h)$, there exists an integer ζ ($1 \leq \zeta \leq d$) such that $\sigma_\zeta(\|\boldsymbol{\theta}^{\mathbf{j}}\|) = \sigma(d, \|\boldsymbol{\theta}^{\mathbf{j}}\|)$. Put

$$\beta(\mathbf{k}, \mathbf{j}) = (\beta(k, j_1), \cdots, \beta(k, j_N)) = \frac{1}{\sqrt{N}M}(\theta^{k-1-j_1}, \cdots, \theta^{k-1-j_N})$$

for sufficiently large $M>0$. Then

$$J_1 \ge \liminf_{|k|\to\infty} \inf_{\underline{j}\le K} \max_{\mathbf{1}\le \boldsymbol{l}\le \beta(\mathbf{k},\mathbf{j})} \frac{X_\zeta(M\boldsymbol{l}\boldsymbol{\theta}^{\mathbf{j}}+\boldsymbol{\theta}^{\mathbf{j}})-X_\zeta(M\boldsymbol{l}\boldsymbol{\theta}^{\mathbf{j}})}{\sigma_\zeta(\|\boldsymbol{\theta}^{\mathbf{j}}\|)\sqrt{2\,\log\big(\Pi_{i=1}^N\beta(k,j_i)\big)}}. \tag{2.23}$$

Let

$$Z_{\mathbf{j}}(\boldsymbol{l}) = \frac{X_\zeta(M\boldsymbol{l}\boldsymbol{\theta}^{\mathbf{j}}+\boldsymbol{\theta}^{\mathbf{j}})-X_\zeta(M\boldsymbol{l}\boldsymbol{\theta}^{\mathbf{j}})}{\sigma_\zeta(\|\boldsymbol{\theta}^{\mathbf{j}}\|)}, \quad \mathbf{1}\le \boldsymbol{l}\le \beta(\mathbf{k},\mathbf{j}).$$

Similarly to (2.10), we have, for all $\boldsymbol{l}$ and $\boldsymbol{l}'$ with $\boldsymbol{l}>\boldsymbol{l}'$,

$$\begin{aligned}\lambda_{\mathbf{j}}(\boldsymbol{l},\boldsymbol{l}') &:= Cov(Z_{\mathbf{j}}(\boldsymbol{l}),Z_{\mathbf{j}}(\boldsymbol{l}'))\\ &= \frac{1}{2\sigma_\zeta^2(\|\boldsymbol{\theta}^{\mathbf{j}}\|)}\Big\{\sigma_\zeta^2(\|M(\boldsymbol{l}-\boldsymbol{l}')\boldsymbol{\theta}^{\mathbf{j}}+\boldsymbol{\theta}^{\mathbf{j}}\|)-\sigma_\zeta^2(\|M(\boldsymbol{l}-\boldsymbol{l}')\boldsymbol{\theta}^{\mathbf{j}}\|)\\ &\qquad -\big(\sigma_\zeta^2(\|M(\boldsymbol{l}-\boldsymbol{l}')\boldsymbol{\theta}^{\mathbf{j}}\|)-\sigma_\zeta^2(\|M(\boldsymbol{l}-\boldsymbol{l}')\boldsymbol{\theta}^{\mathbf{j}}-\boldsymbol{\theta}^{\mathbf{j}}\|)\big)\Big\}.\end{aligned} \tag{2.24}$$

If the right hand side of (2.24) is less than or equal to zero, then it follows from Lemma 2.8 that, for any $0<\varepsilon<1$,

$$\begin{aligned}P\Big\{\inf_{\underline{j}\le K} \max_{\mathbf{1}\le \boldsymbol{l}\le \beta(\mathbf{k}.\mathbf{j})} \frac{Z_{\mathbf{j}}(\boldsymbol{l})}{\sqrt{2\,\log\big(\Pi_{i=1}^N\beta(k,j_i)\big)}} \le \sqrt{1-\varepsilon}\Big\}\\ \le \sum_{\underline{j}\le K}\Big\{\Phi\Big(\sqrt{(2-2\varepsilon)\log\big(\Pi_{i=1}^N\beta(k,j_i)\big)}\Big)\Big\}^{\Pi_{i=1}^N\beta(k,j_i)}.\end{aligned} \tag{2.25}$$

On the other hand, if the right hand side of (2.24) is positive, that is, σ_ζ^2 is a nearly convex function, then it follows from the regular variation of σ_ζ^2 and Lemma 2.7 with $\mathbf{a}=\boldsymbol{\theta}^{\mathbf{j}}$ and $\mathbf{b}=M(\boldsymbol{l}-\boldsymbol{l}')$ that

$$\begin{aligned}|\lambda_{\mathbf{j}}(\boldsymbol{l},\boldsymbol{l}')| &\le \frac{1}{\sigma_\zeta^2(\|\boldsymbol{\theta}^{\mathbf{j}}\|)}\bigg|\int_{\|\boldsymbol{\theta}^{\mathbf{j}}\|\,\|M(\boldsymbol{l}-\boldsymbol{l}')\|}^{\|\boldsymbol{\theta}^{\mathbf{j}}\|\,\|M(\boldsymbol{l}-\boldsymbol{l}')+\mathbf{1}\|} d\sigma_\zeta^2(x) - \int_{\|\boldsymbol{\theta}^{\mathbf{j}}\|\,\|M(\boldsymbol{l}-\boldsymbol{l}')-\mathbf{1}\|}^{\|\boldsymbol{\theta}^{\mathbf{j}}\|\,\|M(\boldsymbol{l}-\boldsymbol{l}')\|} d\sigma_\zeta^2(x)\bigg|\\ &\le C\,\frac{\sigma_\zeta^2(\|\boldsymbol{\theta}^{\mathbf{j}}\|\,\|M(\boldsymbol{l}-\boldsymbol{l}')+\mathbf{1}\|)}{\sigma_\zeta^2(\|\boldsymbol{\theta}^{\mathbf{j}}\|)\,\|M(\boldsymbol{l}-\boldsymbol{l}')-\mathbf{1}\|^2}\\ &\le C\,\frac{\|M(\boldsymbol{l}-\boldsymbol{l}')+\mathbf{1}\|^2}{\|M(\boldsymbol{l}-\boldsymbol{l}')-\mathbf{1}\|^2}\,\|M(\boldsymbol{l}-\boldsymbol{l}')+\mathbf{1}\|^{2\alpha-2}\\ &< \xi\,\|\boldsymbol{l}-\boldsymbol{l}'\|^{-\nu},\end{aligned}$$

for sufficiently small $\xi > 0$, where $\nu = 1 - \alpha > 0$. Let us apply Lemmas 2.9 and 2.10 for

$$\begin{aligned}
&Y(l_{\boldsymbol{l}}) = Z_{\mathbf{j}}(\boldsymbol{l}), \quad \mathbf{1} \le \boldsymbol{l} \le \beta(\mathbf{k}.\mathbf{j}), \quad \mathbf{m} = \beta(\mathbf{k}, \mathbf{j}), \\
&|\lambda(\boldsymbol{l}, \boldsymbol{l}')| = |\lambda_{\mathbf{j}}(\boldsymbol{l}, \boldsymbol{l}')| < \xi \, \|\boldsymbol{l} - \boldsymbol{l}'\|^{-\nu}, \quad \nu = 1 - \alpha > 0, \\
&u = \{(2-\eta) \log\bigl(\Pi_{i=1}^N \beta(k, j_i)\bigr)\}^{1/2}, \quad \eta = 2\varepsilon.
\end{aligned}$$

Then we have

$$\begin{aligned}
&P\Biggl\{ \inf_{\underline{j} \le K} \max_{\mathbf{1} \le \boldsymbol{l} \le \beta(\mathbf{k}.\mathbf{j})} \frac{Z_{\mathbf{j}}(\boldsymbol{l})}{\sqrt{2 \, \log\bigl(\Pi_{i=1}^N \beta(k, j_i)\bigr)}} \le \sqrt{1 - \varepsilon} \Biggr\} \\
&\le \sum_{\underline{j} \le K} \Bigl\{ \bigl(\Phi(u)\bigr)^{\Pi_{i=1}^N \beta(k, j_i)} + c \Bigl(\prod_{i=1}^N \beta(k, j_i) \Bigr)^{-\delta_0} \Bigr\} \\
&\le \sum_{\underline{j} \le K} \Bigl\{ \exp\bigl(- c\, \theta^{\varepsilon N(k - \underline{j})} \bigr) + c \, \bigl(\theta^{N(k - \underline{j})}\bigr)^{-\delta_0} \Bigr\} \\
&\le c \sum_{\underline{j} \le K} \theta^{-N\delta_0(k - \underline{j})} \le c\, \theta^{-N\delta_0 \gamma (\log_\theta \log \theta^{|k|}) / \log \theta} \\
&\le c\, |k|^{-N\delta_0\gamma / \log\theta}
\end{aligned} \tag{2.26}$$

for sufficiently large $|k|$. Note that the right hand side of (2.25) is less than or equal to that of (2.26). Taking $\theta > 1$ such that $\log\theta < N\delta_0\gamma$ in (2.26), then the Borel-Cantelli lemma implies (2.22) via (2.23).

Next, we turn to show that

$$J_2 \le 2c\, \varepsilon^{\alpha/2} \qquad \text{a.s.} \tag{2.27}$$

for any small $\varepsilon > 0$, where $c > 0$ is a constant. Since $\sigma(d, h)$ is regularly varying, we have

$$\frac{\sigma(d, \|\boldsymbol{\theta}^{\mathbf{j}} - \boldsymbol{\theta}^{\mathbf{j-1}}\|)}{\sigma(d, \|\boldsymbol{\theta}^{\mathbf{j-1}}\|)} \le c\, \varepsilon^{\alpha/2}.$$

Therefore, (2.27) is proved if we show that

$$\limsup_{|k| \to \infty} \sup_{\underline{j} \le K} \sup_{\|\mathbf{t}\| \le \theta^k} \sup_{\boldsymbol{\theta}^{\mathbf{j}-1} \le \mathbf{s} \le \boldsymbol{\theta}^{\mathbf{j}}} \frac{\|X^d(\mathbf{t} + \boldsymbol{\theta}^{\mathbf{j}}) - X^d(\mathbf{t} + \mathbf{s})\|}{\sigma(d, \|\boldsymbol{\theta}^{\mathbf{j}} - \boldsymbol{\theta}^{\mathbf{j-1}}\|) \sqrt{2N \log \theta^{k - \underline{j} + 1}}} \le 2 \quad \text{a.s.} \tag{2.28}$$

Applying the same way as the proof of Lemma 2.1, then it follows that, for sufficiently large k,

$$\begin{aligned}
&P\left\{\sup_{\|\mathbf{t}\|\le\theta^k}\ \sup_{\boldsymbol{\theta}^{\mathbf{j-1}}\le\mathbf{s}\le\boldsymbol{\theta}^{\mathbf{j}}}\frac{\|X^d(\mathbf{t}+\boldsymbol{\theta}^{\mathbf{j}})-X^d(\mathbf{t}+\mathbf{s})\|}{\sigma(d,\|\boldsymbol{\theta}^{\mathbf{j}}-\boldsymbol{\theta}^{\mathbf{j-1}}\|)\sqrt{2N\log\theta^{k-\underline{j}+1}}}\ge 2+\varepsilon\right\}\\
&\le c\,\frac{\theta^{Nk}}{\|\boldsymbol{\theta}^{\mathbf{j}}-\boldsymbol{\theta}^{\mathbf{j-1}}\|^N}\exp\left(-\frac{4(2+\varepsilon)^2}{(2+\varepsilon)^2}N\log\theta^{k-\underline{j}+1}\right)\\
&\le c\,(\theta^{3N})^{-(k-\underline{j})}.
\end{aligned}$$

Since

$$\sum_{|k|=1}^{\infty}\sum_{\underline{j}\le K}(\theta^{3N})^{-(k-\underline{j})}\le c\sum_{|k|=1}^{\infty}|k|^{-\gamma/\log\theta}<\infty,$$

we obtain (2.28) and hence (2.20) holds true by (2.27), (2.22) and (2.21). $\square$

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