Strong invariance principles for sequential Bahadur-Kiefer and Vervaat error processes of long-range dependent sequences

Miklós Csörgő,[1] Barbara Szyszkowicz[1] and Lihong Wang[2]

[1] *School of Mathematics and Statistics, Carleton University*
1125 Colonel By Drive, Ottawa, ON, K1S 5B6 Canada
[2] *Department of Mathematics, Nanjing University*
Nanjing, 210093, P.R. China

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Abstract

In this paper we study strong approximations (invariance principles) of the sequential uniform and general Bahadur-Kiefer processes of long-range dependent sequences. We also investigate the strong and weak asymptotic behavior of the sequential Vervaat process, i.e., the integrated sequential Bahadur-Kiefer process, properly normalized, as well as that of its deviation from its limiting process, the so-called Vervaat error process. It is well known that the Bahadur-Kiefer and the Vervaat error processes cannot converge weakly in the i.i.d. case. In contrast to this we conclude that the Bahadur-Kiefer and Vervaat error processes, as well as their sequential versions, do converge weakly to a Dehling-Taqqu type limit process for certain long-range dependent sequences.

1 Introduction

Assume that we have a stationary long-range dependent sequence of standard Gaussian random variables, $\eta_1, \eta_2, \cdots, \eta_n, \cdots$, i.e., the Gaussian sequence $\{\eta_n, n \geq 1\}$ with $\mathrm{E}\eta_1 = 0$ and $\mathrm{E}\eta_1^2 = 1$ is assumed to have a positive covariance function of the form

$$\gamma(k) := \mathrm{E}(\eta_1 \eta_{k+1}) = k^{-D} L(k), \quad 0 < D < 1, \tag{1.1}$$

for large k, where $L(\cdot)$ is a slowly varying function at infinity in the sense that

$$\lim_{s \to \infty} \frac{L(st)}{L(s)} = 1 \quad \text{for every} \quad t \in (0, \infty).$$

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Let G be an arbitrary real-valued Borel measurable function on the real line $\mathbf{R}$, and consider the subordinate process

$$X_n = G(\eta_n), \quad n \geq 1, \tag{1.2}$$

with marginal distribution function $F(x) = P(X \leq x)$, $x \in \mathbf{R}$, where $X = G(\eta)$ and η is a standard normal random variable.

The assumption (1.2) allows one to use the theory of nonlinear functionals of Gaussian processes. As in Dehling and Taqqu (1989), we expand the function $I(X_n \leq x) - F(x) = I(G(\cdot) \leq x) - F(x)$ in Hermite polynomials, for any fixed $x \in \mathbf{R}$,

$$I(X_n \leq x) - F(x) = \sum_{l=\tau_x}^{\infty} c_l(x) H_l(\eta_i)/l!,$$

where

$$H_l(x) = (-1)^l e^{\frac{x^2}{2}} \frac{d^l}{dx^l} e^{-\frac{x^2}{2}}, \quad l = 1, 2, \cdots, \quad x \in \mathbf{R},$$

is the l-th Hermite polynomial,

$$c_l(x) = \mathrm{E}\left\{\left[I\left(G(\eta) \leq x\right) - F(x)\right] H_l(\eta)\right\},$$

and τ_x for any $x \in \mathbf{R}$ is the index of the first nonzero coefficient in the expansion, and it is called the Hermite rank of the function $I(G(\cdot) \leq x) - F(x)$. Then, as in Dehling and Taqqu (1989), the Hermite rank of the class of functions $\{I(X_n \leq x) - F(x), x \in \mathbf{R}\}$ is defined by

$$\tau = \min\{\tau_x : c_{\tau_x}(x) \neq 0 \quad \text{for some} \quad x \in \mathbf{R}\}, \tag{1.3}$$

i.e., $\tau = \inf_x \tau_x$. If we assume that F is continuous, then the induced sequence of random variables

$$U_n = F(X_n) = F(G(\eta_n)), \quad n \geq 1, \tag{1.4}$$

is a Uniform-$[0, 1]$ random sequence. Consequently, for any fixed $y \in (0, 1)$, the function $(I(U_n \leq y) - y) = (I(F(G(\cdot)) \leq y) - y)$ has the Hermite expansion

$$I(U_n \leq y) - y = \sum_{l=\tau}^{\infty} J_l(y) H_l(\eta_n)/l!,$$

where

$$J_l(y) = \mathrm{E}\{[I(F(G(\eta)) \le y) - y]H_l(\eta)\}.$$

Obviously, $J_l(y) = c_l(Q(y))$ for any $y \in (0,1)$, where Q is the quantile function of F, i.e.,

$$Q(y) = F^{-1}(y) = \inf\{x : F(x) = y\}, 0 < y \le 1, \quad Q(0) = Q(0+),$$

and hence the Hermite rank of the class of functions $\{I(U_n \le y) - y, y \in (0,1)\}$ is also τ.

Given chronologically ordered samples $X_1, \cdots, X_n$ and $U_1, \cdots, U_n$, $n \ge 1$, as in (1.2) and (1.4) respectively, their corresponding sequential empirical distribution functions are

$$\widehat{E_{[nt]}}(y) = \begin{cases} 0, & 0 \le t < 1/n, \\ \frac{1}{[nt]} \sum_{i=1}^{[nt]} I(U_i \le y), & 0 \le y \le 1, 1/n \le t \le 1, \end{cases}$$

and

$$\widehat{F_{[nt]}}(x) = \begin{cases} 0, & 0 \le t < 1/n, \\ \frac{1}{[nt]} \sum_{i=1}^{[nt]} I(X_i \le x), & -\infty < x < \infty, 1/n \le t \le 1. \end{cases}$$

Based on these functions, we define the sequential empirical quantile functions

$$\widehat{U_{[nt]}}(y) = \widehat{E_{[nt]}}^{-1}(y) = \inf\{s : \widehat{E_{[nt]}}(s) \ge y\}, 0 < y \le 1,$$

$$\widehat{U_{[nt]}}(0) = \widehat{U_{[nt]}}(0+), \quad 0 \le t \le 1,$$

and

$$\widehat{Q_{[nt]}}(y) = \widehat{F_{[nt]}}^{-1}(y) = \inf\{x : \widehat{F_{[nt]}}(x) \ge y\}, 0 < y \le 1,$$

$$\widehat{Q_{[nt]}}(0) = \widehat{Q_{[nt]}}(0+), \quad 0 \le t \le 1.$$

Now the corresponding sequential uniform and general empirical and quantile processes are defined by

$$\alpha_n(y,t) = d_n^{-1}[nt](\widehat{E_{[nt]}}(y) - y), \quad 0 \le y \le 1, 0 \le t \le 1,$$

$$u_n(y,t) = d_n^{-1}[nt](y - \widehat{U_{[nt]}}(y)), \quad 0 \le y \le 1, 0 \le t \le 1,$$

$$\beta_n(x,t) = d_n^{-1}[nt](\widehat{F_{[nt]}}(x) - F(x)), \quad -\infty < x < \infty, 0 \le t \le 1,$$

$$\gamma_n(y,t) = d_n^{-1}[nt](Q(y) - \widehat{Q_{[nt]}}(y)), \quad 0 < y < 1, 0 \le t \le 1,$$

where

$$d_n^2 = n^{2-\tau D} L^\tau(n) \tag{1.5}$$

with $0 < D < 1/\tau$, where τ is defined in (1.3).

By Theorem 3.1 of Taqqu (1975) one arrives at

$$\mathrm{Var}(n\widehat{F_n}(x)) \sim n^{2-\tau D} L^\tau(n) \frac{2c_\tau^2(x)}{\tau!(2-\tau D)(1-\tau D)} = O(d_n^2)$$

for each fixed $x \in \mathrm{R}$ as $n \to \infty$, where the symbol $\sim$ means asymptotic proportional equivalence. This explains the choice of d_n as defined in (1.5) for defining the above sequential empirical and quantile processes.

Dehling and Taqqu (1988, 1989) studied the asymptotic properties of the sequential general empirical process $\beta_n(x,t)$. The following important two-parameter weak convergence theorem for $\beta_n(x,t)$ is due to Dehling and Taqqu (1989) whose Theorem 1.1 reads as follows.

Theorem A. *Let the stationary subordinate process* $\{X_n, n \ge 1\}$ *be as in* (1.2) *with* τ *as in* (1.3), *and let* d_n *be as in* (1.5). *Then, as* $n \to \infty$,

$$\{\beta_n(x,t); -\infty \le x \le +\infty, 0 \le t \le 1\} \quad \textit{converges weakly in} \quad D[-\infty,+\infty] \times [0,1],$$

equipped with the sup-norm, to

$$\left\{ c_\tau(x) \sqrt{\frac{2}{(2-\tau D)(1-\tau D)}} Y_\tau(t); -\infty \le x \le +\infty, 0 \le t \le 1 \right\}, \quad 0 < D < 1/\tau,$$

where $Y_\tau(t)$ *is* $1/\tau!$ *times a Hermite process of rank* τ, *given for each* $t \in [0,1]$ *as a multiple Wiener-Itô-Dobrushin integral that is defined in* (1.7) *of* Dehling and Taqqu (1989).

We also note that Dehling and Taqqu (1988) obtained the functional law of the iterated logarithm as well for $\beta_n(x,t)$ in $D[-\infty,+\infty] \times [0,1]$.

Remark 1.1. We recall (cf. Müller (1970)) that in the i.i.d. case the weak limit of $\beta_n(x,t)$ is a two-time parameter Gaussian process in x and t, the so-called Kiefer process on account of the landmark Kiefer (1972) paper, which is a Brownian bridge in x and a Wiener process (Brownian motion) in t. The Dehling-Taqqu (1989) limit in Theorem A differs greatly from the Kiefer process. Namely, it separates the variables in x and t in terms of being the product of a *deterministic function* in x and a *stochastic process* in t which is non-Gaussian when $\tau \geq 2$.

Assuming that F has a Lebesgue density function f on $\mathbf{R}$, S. Csörgő and Mielniczuk (1995) showed that the kernel estimators based density process corresponding to the general empirical process $\beta_n(x,1)$ converges weakly with the same normalization to the derivative of the limiting process in Theorem 1.1 of Dehling and Taqqu (1989) that we quoted as Theorem A here.

We note that, with F continuous, we have

$$\alpha_n(y,t) = \beta_n(Q(y),t), \quad y,t \in [0,1], \quad \text{and} \quad \beta_n(x,t) = \alpha_n(F(x),t), \quad x \in \mathbf{R}, t \in [0,1].$$

Hence, if F is continuous, all strong and weak asymptotic results hold true simultaneously for both $\beta_n(x,t)$ and $\alpha_n(y,t)$.

For further reference we spell out the weak convergence result that follows from Theorem A for $\alpha_n(y,t) = \beta_n(Q(y),t)$, $y,t \in [0,1]$, based on the induced sequence $\{U_n, n \geq 1\}$ as in (1.4).

Corollary A. *With F continuous and τ and D as in* (1.3) *and* (1.5) *respectively, as $n \to \infty$, we have*

$$\begin{aligned}\alpha_n(y,t) = \beta_n(Q(y),t) \quad &\xrightarrow{\mathcal{D}} \sqrt{\tfrac{2}{(2-\tau D)(1-\tau D)}}\, c_\tau(Q(y)) Y_\tau(t) \\ &= \sqrt{\tfrac{2}{(2-\tau D)(1-\tau D)}}\, J_\tau(y) Y_\tau(t)\end{aligned}$$

in $D[0,1]^2$ that is equipped with sup-norm, where, as before, $Y_\tau(t)$ is $1/\tau!$ times a Hermite process of rank τ, given for each $t \in [0,1]$ as a multiple Wiener-Itô-Dobrushin integral as in Theorem A.

In this paper we go further along these lines and establish strong approximations of the sequential uniform and general quantile processes, and of the sequential Bahadur-Kiefer processes as defined in (1.7) and (1.8) below. Moreover, we also study the sequential uniform Vervaat and Vervaat error processes of (1.9) and (1.10) respectively, along the same lines.

Since there is no simple relationship between $u_n(y,t)$ and $\gamma_n(y,t)$, following Csörgő and Révész (1978) in the i.i.d. case along the lines of Csörgő and Szyszkowicz (1998), here too we shall consider the normalized sequential general quantile process

$$\begin{aligned}\rho_n(y,t) = f(Q(y)\gamma_n(y,t) \quad &= d_n^{-1}[nt] f(Q(y))(Q(y) - \widehat{Q_{[nt]}}(y)) \\ &= u_n(y,t)\tfrac{f(Q(y))}{f(Q(\theta_n(y,t)))},\end{aligned} \tag{1.6}$$

where $0 \le y,t \le 1$, $|y - \theta_n(y,t)| \le |y - \widehat{U_{[nt]}}(y)|$, provided that F is an absolutely continuous distribution function with a strictly positive Lebesgue density function f on the real line.

We define the stochastic processes

$$\begin{aligned}&\{R_n^*(y,t), 0 \le y \le 1, 0 \le t \le 1, n = 1,2\cdots\} \\ &= \{d_n(\alpha_n(y,t) - u_n(y,t)), 0 \le y \le 1, 0 \le t \le 1, n = 1,2\cdots\},\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}&\{R_n(y,t), 0 \le y \le 1, 0 \le t \le 1, n = 1,2\cdots\} \\ &= \{d_n(\alpha_n(y,t) - \rho_n(y,t)), 0 \le y \le 1, 0 \le t \le 1, n = 1,2\cdots\} \\ &= \{d_n(\beta_n(Q(y),t) - \rho_n(y,t)), 0 \le y \le 1, 0 \le t \le 1, n = 1,2\cdots\},\end{aligned} \tag{1.8}$$

which rhyme with the sequential uniform and general Bahadur-Kiefer processes respectively in the i.i.d. case that enjoy some remarkable asymptotic properties (cf. Bahadur (1966), Kiefer (1967, 1970)). For a review of various aspects of this subject in the i.i.d. case we refer to Csörgő and Révész (1981), Csörgő (1983), Shorack and Wellner (1986), Csörgő and Szyszkowicz (1998), Csáki et al. (2002), Csörgő and Zitikis (2002), and the references therein.

One of the remarkable asymptotic properties of the sequential Bahadur-Kiefer process in the i.i.d. case is that $a_n R_n^*(\cdot,\cdot)$ cannot converge weakly in the space $D[0,1]^2$ for any normalizing sequence $\{a_n\}$ of positive real numbers (cf. Vervaat (1972a,b), and Csáki et al. (2002) for a review of this matter in case of $R_n^*(\cdot,1)$).

On the other hand, with $d_n = n^{1/2}$, Vervaat (1972a,b) in the i.i.d. case established the weak convergence of the following integrated Bahadur-Kiefer process

$$V_n(s,t) = 2d_n^{-2}[nt]\int_0^s R_n^*(y,t)dy, \quad 0 \le s \le 1, 0 \le t \le 1, \tag{1.9}$$

the so-called sequential uniform Vervaat process in the case of $t = 1$, via that of $\alpha_n^2(s,1)$, as a consequence of showing that $\sup_{0\le s\le 1}|V_n(s,1) - \alpha_n^2(s,1)| = o_P(1)$, as $n \to \infty$. We define the sequential Vervaat error process $Q_n(s,t)$ by

$$Q_n(s,t) = V_n(s,t) - \alpha_n^2(s,t), \quad 0 \le s \le 1, 0 \le t \le 1. \tag{1.10}$$

Csörgő and Zitikis (2001), Csáki et al. (2002) concluded that, just like the uniform Bahadur-Kiefer process, in the i.i.d. case $a_n Q_n(\cdot, 1)$ cannot converge weakly in the space $D[0,1]$ for any sequence $\{a_n\}$ of positive real numbers. Hence they studied the strong and weak asymptotic point-wise, sup- and L_p-norm behavior of the process $Q_n(\cdot, 1)$ for i.i.d. random samples *à la* Kiefer (1970) with $d_n = n^{1/2}$.

We shall see in this paper that, unlike in the i.i.d. case, when appropriately normalized, the sequential Bahadur-Kiefer processes and the sequential uniform Vervaat error process of long-range dependent sequences as in (1.2) and (1.4) both converge weakly in $D[0,1]^2$, via first establishing strong approximations for these processes in sup-norm. This new phenomenon in this context will be seen to be due to the limiting processes being Dehling-Taqqu type processes (cf. Remark 1.1), i.e., multiplications of a non-random function with a random process which typically is a power of $Y_\tau(t)$, the Hermite process of rank τ of Theorem A. Thus, via strong invariance, we arrive at functional limit theorems and laws of the iterated logarithm for the sequential Bahadur-Kiefer and the sequential uniform Vervaat error processes.

In Sections 2 and 3 we present strong invariance principles (approximations) for the sequential uniform Bahadur-Kiefer process and sequential uniform Vervaat error process of long-range dependent sequences as in (1.2) and (1.4), namely for $R_n^*(y,t)$ and $Q_n(s,t)$ as in (1.7) and (1.10) respectively. Section 4 is devoted to establishing analogous statements for the sequential general Bahadur-Kiefer process $R_n(y,t)$ of (1.8) by examining the sup-norm distance between the sequential uniform quantile process $u_n(y,t)$ and the normalized sequential general quantile process $\rho_n(y,t)$ *à la* Csörgő and Révész (1978), and Csörgő and Szyszkowicz (1998). The results obtained in this paper for long-range dependent sequences are analogs of those in the i.i.d. case in Csörgő and Szyszkowicz (1998), Csörgő and Shi (1998, 2001), Csörgő and Zitikis (1999, 2001), and Csáki et al. (2002).

For a thorough analysis and use of long-range dependence in general, we refer to Beran (1992, 1994), and Doukhan, Oppenheim and Taqqu (2003).

2 Sequential uniform Bahadur-Kiefer process, Strong approximations

2.1 Preliminaries

Throughout this paper we assume that $\{X_n = G(\eta_n)\}$ and $\{U_n = F(G(\eta_n))\}$, $n \geq 1$, are as in (1.2) and (1.4) respectively, long-range dependent random sequences that are governed by the standard Gaussian random process $\{\eta_n\}$ which satisfies (1.1).

We first derive a strong approximation of the sequential general empirical process $\beta_n(x,t)$ by the process $c_\tau(x)\sum_{i=1}^{[nt]} H_\tau(\eta_i)/\tau!$, via changing the rate of convergence in Theorem 3.1 of Dehling and Taqqu (1989)(written as DT (1989) from now on) to fit our purposes in this exposition.

Proposition 2.1 *Let p be the smallest integer satisfying* $\max\left(2, \tau, \frac{\tau D}{1-\tau D}\right) < p \leq \max\left(\frac{4-\tau D}{D}, \frac{4-\tau D}{1-\tau D}\right)$. *Assume that* $\sup_{u\geq 1}\gamma(u) < \delta$, *where* $0<\delta<(p-1)^{-1}$ *and* $\gamma(\cdot)$ *is as in* (1.1). *Then, as* $n\to\infty$, *we have*

$$\sup_{0\leq t\leq 1}\ \sup_{-\infty<x<+\infty} |\beta_n(x,t) - d_n^{-1}c_\tau(x)\sum_{i=1}^{[nt]} H_\tau(\eta_i)/\tau!| = O(n^{-\nu p/2+\tau D/4+\varepsilon}) \quad a.s.$$

with any sufficiently small positive ε, *where* $\nu = \min(D, 1-\tau D)/2$.

Proof. The proof is based on the well-known chaining argument of DT (1989). Hence, while studying their proof of Theorem 3.1 in DT (1989), we shall only briefly indicate the extra steps that are needed for us to achieve our goal.

Let $S_n(k;x,y) = S_n(k;x) - S_n(k;y)$ $(-\infty < y \le x < +\infty)$, where

$$S_n(k;x) = d_n^{-1} \sum_{i=1}^{k} \{I(X_i \le x) - F(x) - c_\tau(x) H_\tau(\eta_i)/\tau!\}, \quad 1 \le k \le n$$

Then we have

$$d_n S_n(k;x,y) = \sum_{i=1}^{k} \sum_{q=\tau+1}^{\infty} \frac{c_q(x) - c_q(y)}{q!} H_q(\eta_i) \le C \sum_{i=1}^{k} \sum_{q=\tau+1}^{\infty} H_q(\eta_i)$$

with some finite constant C.

Via Proposition 4.2 of Taqqu (1977), one can verify that

$$\mathrm{E}|d_n S_n(k;x,y)|^p \le C \left\{ k \sum_{u=0}^{k} |\gamma(u)|^{\tau+1} \right\}^{p/2}.$$

Suppose first $0 < D < (\tau+1)^{-1}$. Then, by (1.1), as $k \to \infty$,

$$k \sum_{u=0}^{k} |\gamma(u)|^{\tau+1} = O(k^{2-(\tau+1)D} L^{\tau+1}(k)).$$

When $D \ge (\tau+1)^{-1}$, $\sum_{u=0}^{k} |\gamma(u)|^{\tau+1}$ is slowing varying as $k \to \infty$, and hence

$$k \sum_{u=0}^{k} |\gamma(u)|^{\tau+1} = O(k L_0(k))$$

for some slowly varying function $L_0(\cdot)$ at infinity. Thus we arrive at

$$\mathrm{E}|S_n(k;x,y)|^p \le C k^{-\nu p+\varepsilon} \left(\frac{d_k}{d_n}\right)^p \le C \left(\frac{k}{n}\right)^{(1-\nu-\tau D/2)p} n^{-\nu p+\varepsilon} \tag{2.1}$$

with any sufficiently small positive ε for any $-\infty < y \le x < +\infty$, $1 \le k \le n$.

For any $s \ge 1$, define the partition

$$-\infty = \pi_{0,s} < \pi_{1,s} < \cdots < \pi_{2^s,s} = +\infty.$$

Given $\zeta > 0$, let $K = [\log_2(C\zeta^{-1} n d_n^{-1})] + 1$. Next, for any $x \in \mathbf{R}$ and $s = 0, 1, \cdots, K$, define j_s^x by

$$\pi_{j_s^x,s} \le y < \pi_{j_s^x+1,s}.$$

One can then define a chain linking $-\infty$ to each point x by

$$-\infty = \pi_{j_0^x,0} \le \pi_{j_1^x,1} \le \cdots \le x < \pi_{j_K^x+1,K}.$$

Now using (2.1) instead of Lemma 3.1 of DT (1989) and applying Chebyshev's inequality, along the same lines as those of the proof of Lemma 3.2 of DT (1989), we obtain

$$\begin{aligned}
& \mathrm{P}\left\{\sup_{-\infty<x<+\infty} |S_n(k;x)| > \zeta\right\} \\
\le\ & \textstyle\sum_{s=0}^{K} \mathrm{P}\left\{\sup_{-\infty<x<+\infty} |S_n(k;\pi_{j_s^x,s},\pi_{j_{s+1}^x,s+1})| > \zeta/(s+3)^2\right\} \\
+\ & \mathrm{P}\left\{d_n^{-1}|\sum_{i=1}^{n} H_\tau(\eta_i)| > 2^{K-1}\zeta/4\right\} \\
\le\ & C(\tfrac{k}{n})^{(1-\nu-\tau D/2)p} n^{-\nu p+\varepsilon}\zeta^{-p} \textstyle\sum_{s=0}^{K} 2^{s+1}(s+3)^{2p} + C(\tfrac{d_k}{d_n})^p \zeta^{-p} 2^{-p(K-1)} \\
\le\ & C(\tfrac{k}{n})^{(1-\nu-\tau D/2)p} n^{-\nu p+\varepsilon}\zeta^{-p} 2^K (K+3)^{2p+1} + C(\tfrac{k}{n})^{(1-\tau D/2)p} n^{-\tau Dp/2+\varepsilon} \\
\le\ & Cn^{-\nu p+\tau D/2+\varepsilon}\left((\tfrac{k}{n})^{(1-\nu-\tau D/2)p}\zeta^{-p-\varepsilon} + (\tfrac{k}{n})^{(1-\tau D/2)p}\right) \\
\le\ & Cn^{-\nu p+\tau D/2+\varepsilon}\left(\zeta^{-p-\varepsilon} + (\tfrac{k}{n})^{(1-\tau D/2)p}\right)
\end{aligned}$$

for any $\zeta \in (0,1]$. The last inequality is due to the fact that $(1-\nu-\tau D/2)p > 1$.

On applying this conclusion, an appropriate variant of the proof of Theorem 3.1 of DT (1989) leads to

$$\mathrm{P}\left\{\max_{k\le n} \sup_{-\infty<x<+\infty} |S_n(k;x)| > \zeta\right\} \le Cn^{-\nu p+\tau D/2+\varepsilon}(1+\zeta^{-p-\varepsilon}). \tag{2.2}$$

We now make use of (2.2) with $n = n_l = \min\{j : j \ge e^l\}$ and $\zeta = \zeta_l = \exp\{l(-\nu p/2 + \tau D/4+\varepsilon)/(p+\varepsilon)\}$, $l = 0, 1, \cdots$. Then, by Borel-Cantelli lemma, there exists an integer l_0 such that for any $l \ge l_0$,

$$\max_{k\le n_l} \sup_{-\infty<x<+\infty} |S_{n_l}(k;x)| \le \exp\{-l(\nu p/2 - \tau D/4 - \varepsilon)\} \quad \text{a.s.}$$

Let $n \ge e^{l_0}$ and let l be the integer such that $n_{l-1} \le n < n_l$. Since $e^{-l} \le n^{-1}$ and $l \to \infty$ as $n \to \infty$, by definition of d_n and that of a slowly varying function, we have

$$\sup_{-\infty<x<+\infty} |S_n(n;x)| \le \frac{d_{n_l}}{d_n} \max_{k\le n_l} \sup_{-\infty<x<+\infty} |S_{n_l}(k;x)| \le Cn^{-\nu p/2+\tau D/4+\varepsilon}.$$

This implies that

$$\sup_{-\infty<x<+\infty} d_n^{-1} n^{\nu p/2-\tau D/4-\varepsilon} |d_n\beta_n(x,1) - c_\tau(x)\sum_{i=1}^{n} H_\tau(\eta_i)/\tau!| = O(1) \quad \text{a.s.}$$

The latter, in turn, gives that, with fixed $t \in (0,1]$ and $(nt) \to \infty$ as $n \to \infty$, we have

$$\begin{aligned}\sup_{-\infty<x<+\infty} \Big|d_n\beta_n(x,t) - c_\tau(x)\sum_{i=1}^{[nt]} H_\tau(\eta_i)/\tau!\ \Big| &= O(d_{[nt]}(nt)^{-\nu p/2+\tau D/4+\varepsilon}) \\ &= O((nt)^{1-\nu p/2-\tau D/4+\varepsilon} L^{\tau/2}(nt)) \quad \text{a.s.,}\end{aligned}$$

and by our assumption for p, we see that the exponent of $(nt)^{1-\nu p/2-\tau D/4+\varepsilon}$ is positive. Hence, without loss of generality, we can assume that the regularly varying function $(nt)^{1-\nu p/2-\tau D/4+\varepsilon} L^{\tau/2}(nt)$ of positive exponent is a strictly monotone increasing regularly function of (nt) (cf. 7. of Corollary 1.2.1 of de Haan (1975), or Theorem 1.5.4 of Bingham, Goldie and Teugels (1987)). Hence, on dividing both sides by $n^{1-\nu p/2-\tau D/4+\varepsilon} L^{\tau/2}(n)$ and then taking $\sup_{0\le t\le 1}$ on both sides, we obtain the result of Proposition 2.1. $\square$

Remark 2.1 Just like Theorem 3.1 of Dehling and Taqqu (1989), Proposition 2.1 implies Theorem A, i.e., Theorem 1.1 of Dehling and Taqqu (1989). Moreover, in case of some important special cases, Proposition 2.1 can be changed into weighted sequential approximations in probability along the lines of Szyszkowicz (1998). For example, if in (1.2) $G(x) = x$, then τ of (1.3) is equal to 1, and $Y_1(t)$ of Theorem A is a fractional Brownian motion with variance t^{2-D}, $0 < D < 1$, (cf. Example 1 of Dehling and Taqqu (1989)). Let now $\widetilde{Q}$ be the class of positive functions q on $(0,1]$, i.e., $\inf_{\delta\le t\le 1} q(t) > 0$ for all $0 < \delta < 1$, for which we have

$$\text{(a)} \quad \lim_{t\downarrow 0} |Y_1(t)|/q(t) = 0 \quad \text{a.s.} \quad \text{or} \quad \text{(b)} \quad \limsup_{t\downarrow 0} |Y_1(t)|/q(t) < \infty \quad \text{a.s.}$$

Then, characterizing the class of functions $\widetilde{Q}$ in cases of (a) and (b) respectively along the lines of M. Csörgő, S. Csörgő, Horváth and Mason (1986), appropriate analogs of the results of Szyszkowicz (1998) in weighted sup-norm and L_p-metrics will continue to hold true in this context as well. Further to this, the rest of this exposition can also be extended along the lines of Section 3 of Csörgő and Szyszkowicz (1998) in the special case of $G(x) = x$, i.e., when $Y_1(t)$ of Theorem A is a fractional Brownian motion, so that, in this special case, in probability and weak convergence versions of our results would hold in weighted sup-norm and L_p-metrics. However, we will not attempt to carry out this program in our present paper.

Remark 2.2 Another interesting special case is $G(x) = x^2$, which gives rise to a class of functions of Hermite rank $\tau = 2$ (cf. Exmaple 2 of Dehling and Taqqu (1989)). Then $Y_2(t)$ of Theorem A is called the Rosenblatt process (cf. Taqqu (1975)). $Y_2(t)$ is non-Gaussian, has stationary increments and the same covariance function as $Y_1(t)$, i.e.,

$$EY_2(s)Y_2(t) = \frac{1}{2}\left\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right\},$$

but with $H = 1 - D$, $0 < D < 1/2$ (in case of $EY_1(s)Y_1(t)$, on the right hand side $H = 1 - D/2$, $0 < D < 1$). Mutatis mutandis, the program that is outlined in Remark 2.1 may also be feasible in terms of $Y_2(t)$, though likely more difficult as well.

In the rest of this paper the marginal distribution function F of $\{X_n\}$ in (1.2) is assumed to be continuous. We also assume

Assumption (A): $J_\tau(y)$ and the derivatives $J'_\tau(y)$, $J_\tau{}''(y)$ with τ as in (1.3) are uniformly bounded and

$$\sup_{0<y\le\delta_n} |J_\tau(y)| = O(\delta_n)$$

for any sequence $\delta_n \to 0$ as $n \to \infty$.

Remark 2.3 Since we assume that F is continuous, $J_\tau(0) = 0$, it follows that

$$\sup_{0<y\le\delta_n} |J_\tau(y)| = \sup_{0<y\le\delta_n} |J_\tau(y) - J_\tau(0)| \le \sup_{0<y\le\delta_n} y \cdot \sup_{0<\theta\le\delta_n} |J'_\tau(\theta)| = O(\delta_n).$$

Moreover, if we take G as $G = F^{-1}\Phi$, we see that $J_1(y) = -\phi(\Phi^{-1}(y)) \ne 0$ for any $y \in (0,1)$, where ϕ, Φ^{-1} denote respectively the density function and the quantile function of the unit normal distribution function Φ. This means that in this case $\tau = 1$, and elementary calculations show that Assumption (A) holds true automatically.

For the sake of first approximating the sequential uniform empirical and quantile processes $\alpha_n(y,t)$ and $u_n(y,t)$, we define the two-time parameter stochastic process

$\{V(y,nt);\, 0 \le y \le 1,\, 0 \le t \le 1, n \ge 1\}$ by

$$V(y,nt) = J_\tau(y) \sum_{i=1}^{[nt]} H_\tau(\eta_i)/\tau!, \tag{2.3}$$

and, as an immediate consequence of Proposition 2.1, we conclude the following strong approximation for the sequential uniform empirical process $\alpha_n(y,t)$.

Corollary 2.1 *Under the assumptions of* Proposition 2.1, *we have*

$$\sup_{0\le t\le 1} \sup_{0\le y\le 1} |\alpha_n(y,t) - d_n^{-1} V(y,nt)| = O(n^{-\nu p/2 + \tau D/4 + \varepsilon}) \quad a.s.$$

with any sufficiently small positive ε, where $\nu = \min(D, 1-\tau D)/2$.

Let $\kappa_{1\tau} = \sup_{0\le y\le 1} |J_\tau(y)|$, $\kappa_{2\tau} = \sup_{0\le y\le 1} |J_\tau(y) \cdot J'_\tau(y)|$, $\kappa_{3\tau} = \sup_{0\le y\le 1} |J^2_\tau(y) \cdot J'_\tau(y)|$. Via Assumption (A) we conclude $0 < \kappa_{1\tau}, \kappa_{2\tau}, \kappa_{3\tau} < \infty$. Moreover, if we take $G = F^{-1}\Phi$, by Remark 2.3 it is easy to check that $\kappa_{11} = 1/(2\pi)^{1/2}$, $\kappa_{21} = 1/(2\pi \mathrm{e})^{1/2}$ and $\kappa_{31} = 1/\{2\pi(2\mathrm{e})^{1/2}\}$.

The process $V(y,nt)$ defined in (2.3) that is approximating $\alpha_n(y,t)$ as in Corollary 2.1 can also be used to approximate the sequential uniform quantile process $u_n(y,t)$. Namely, we have

Proposition 2.2 *Let p be the smallest integer satisfying* $\max\left(3\tau, \frac{3\tau D}{1-\tau D}\right) < p \le \max\left(\frac{4-\tau D}{D}, \frac{4-\tau D}{1-\tau D}\right)$. *Suppose* Assumption (A) *holds. Then under the assumptions of* Corollary 2.1, *as $n \to \infty$, we have*

$$\sup_{0\le t\le 1} \sup_{0\le y\le 1} |u_n(y,t) - d_n^{-1} V(y,nt)| = O(n^{-\tau D/2} L^{\tau/2}(n) (\log\log n)^\tau) \quad a.s. \tag{2.4}$$

Proof. Note that

$$\begin{aligned} u_n(y,t) &= d_n^{-1}[nt]\{\widehat{E_{[nt]}}(\widehat{U_{[nt]}}(y)) - \widehat{U_{[nt]}}(y)\} - d_n^{-1}[nt]\{\widehat{E_{[nt]}}(\widehat{U_{[nt]}}(y)) - y\} \\ &= \alpha_n(\widehat{U_{[nt]}}(y), t) - d_n^{-1}[nt]\{\widehat{E_{[nt]}}(\widehat{U_{[nt]}}(y)) - y\}, \end{aligned}$$

and it is easy to see that

$$0 \le \sup_{0\le y\le 1} |\widehat{E_n}(\widehat{U_{[nt]}}(y)) - y| \le 1/[nt].$$

Thus we have

$$\sup_{0\le t\le 1}\sup_{0\le y\le 1}|u_n(y,t)-\alpha_n(y,t)| = \sup_{0\le t\le 1}\sup_{0\le y\le 1}|\alpha_n(\widehat{U_{[nt]}}(y),t)-\alpha_n(y,t)|+O(d_n^{-1}). \quad (2.5)$$

Applying Corollary 2.1, estimating the right hand side of (2.5) we obtain

$$\begin{aligned}&\sup_{0\le t\le 1}\sup_{0\le y\le 1}|\alpha_n(\widehat{U_{[nt]}}(y),t)-\alpha_n(y,t)|\\ &=\sup_{0\le t\le 1}\sup_{0\le y\le 1}d_n^{-1}|V(\widehat{U_{[nt]}}(y),nt)-V(y,nt)|+O(n^{-\nu p/2+\tau D/4+\varepsilon})\quad \text{a.s.}\end{aligned} \quad (2.6)$$

Hence we need to study the size of the random increments of the process $V(y,nt)$.

The Mori-Oodaira LIL (1987) yields

$$\limsup_{n\to\infty} n^{\tau D/2-1}(L(n)\log\log n)^{-\tau/2}\sup_{0\le t\le 1}|\sum_{i=1}^{[nt]}H_\tau(\eta_i)/\tau!| = \frac{2^{(\tau+1)/2}}{\sqrt{\tau!(2-\tau D)(1-\tau D)}}\quad \text{a.s.} \quad (2.7)$$

Hence, by (2.3) and the fact that $0<\kappa_{1\tau}<\infty$, we have

$$\limsup_{n\to\infty}(\log\log n)^{-\tau/2}\sup_{0\le t\le 1}\sup_{0\le y\le 1}d_n^{-1}|V(y,nt)| = \frac{2^{(\tau+1)/2}\kappa_{1\tau}}{\sqrt{\tau!(2-\tau D)(1-\tau D)}}\quad \text{a.s.} \quad (2.8)$$

Consequently, via Corollary 2.1 we conclude

$$\limsup_{n\to\infty}(\log\log n)^{-\tau/2}\sup_{0\le t\le 1}\sup_{0\le y\le 1}|\alpha_n(y,t)| = \frac{2^{(\tau+1)/2}\kappa_{1\tau}}{\sqrt{\tau!(2-\tau D)(1-\tau D)}}\quad \text{a.s.},$$

and this, in turn, gives

$$\limsup_{n\to\infty}(\log\log n)^{-\tau/2}\sup_{0\le t\le 1}\sup_{0\le y\le 1}|u_n(y,t)| = \frac{2^{(\tau+1)/2}\kappa_{1\tau}}{\sqrt{\tau!(2-\tau D)(1-\tau D)}}\quad \text{a.s.} \quad (2.9)$$

on account of

$$\sup_{0\le t\le 1}\sup_{0\le y\le 1}|\alpha_n(y,t)| = \sup_{0\le t\le 1}\sup_{0\le y\le 1}|u_n(y,t)|.$$

On the other hand, by the mean value theorem we arrive at

$$|J_\tau(\widehat{U_n}(y))-J_\tau(y)| = |\widehat{U_n}(y)-y||J'_\tau(\theta_{1n}(y))|,$$

where $|y-\theta_{1n}(y)|\le|\widehat{U_n}(y)-y|$. Now (2.9) with $t=1$ implies that, as $n\to\infty$,

$$\sup_{0\le y\le 1}|\widehat{U_n}(y)-y| = \sup_{0\le y\le 1}d_n n^{-1}|u_n(y,1)| = O((n^{-D}L(n)\log\log n)^{\tau/2})\to 0 \quad (2.10)$$

almost surely (we note in passing that (2.10) is just a *Glivenko-Cantelli theorem* with rates of convergence in terms of the long-range dependent sequence as in (1.4)). Thus, by Assumption (A), as $n \to \infty$, we arrive at

$$\sup_{0\le y\le 1} |J_\tau(\widehat{U_n}(y)) - J_\tau(y)| = O((n^{-D}L(n)\log\log n)^{\tau/2}) \quad \text{a.s.}$$

The latter combined with (2.7) for $t=1$ yields

$$\sup_{0\le y\le 1} d_n^{-1}|V(\widehat{U_n}(y),n) - V(y,n)| = O(n^{-\tau D/2}L^{\tau/2}(n)(\log\log n)^{\tau}) \quad \text{a.s.} \tag{2.11}$$

Using (2.5)-(2.6), (2.11) and our assumption for p, we arrive at

$$\sup_{0\le y\le 1} |u_n(y,1) - \alpha_n(y,1)| = O(n^{-\tau D/2}L^{\tau/2}(n)(\log\log n)^{\tau}) \quad \text{a.s.} \tag{2.12}$$

Now (2.12) combined with Corollary 2.1 with $t=1$ yields

$$\sup_{0\le y\le 1} |u_n(y,1) - d_n^{-1}V(y,n)| = O(n^{-\tau D/2}L^{\tau/2}(n)(\log\log n)^{\tau}) \quad \text{a.s.}$$

On multiplying through by d_n and then applying a similar argument as used at the end of the proof of Proposition 2.1, we conclude (2.4). $\square$

Next, in view of (2.5) and (2.6) we establish the exact size of the random increments of the process $V(y,nt)$ for convenient use later on.

Proposition 2.3 *Under the assumptions of* Proposition 2.2, *we have*

$$\limsup_{n\to\infty} n^{\tau D-1}(L(n)\log\log n)^{-\tau} \sup_{0\le t\le 1}\sup_{0\le y\le 1} |V(\widehat{U_{[nt]}}(y),nt) - V(y,nt)| = \frac{2^{\tau+1}\kappa_{2\tau}}{\tau!(2-\tau D)(1-\tau D)} \quad a.s.$$

Proof. We note that

$$\begin{aligned} & J_\tau(\widehat{U_n}(y)) - J_\tau(y) \\ = \;& J_\tau'(y)(\widehat{U_n}(y) - y) + \tfrac{1}{2}(\widehat{U_n}(y)-y)^2 J_\tau''(\theta_{2n}(y)) \\ = \;& -J_\tau'(y)n^{-1}V(y,n) + J_\tau'(y)d_n n^{-1}(d_n^{-1}V(y,n) - u_n(y)) + \tfrac{1}{2}(\widehat{U_n}(y)-y)^2 J_\tau''(\theta_{2n}(y)), \end{aligned}$$

where $|y-\theta_{2n}(y)| \le |\widehat{U_n}(y) - y|$. Consequently, by (2.4) with $t=1$ and (2.10) we obtain

$$\sup_{0\le y\le 1} |J_\tau'(y)d_n n^{-1}(u_n(y) - d_n^{-1}V(y,n))| = O((n^{-D}L(n)\log\log n)^{\tau}) \quad \text{a.s.}$$

and

$$\sup_{0\le y\le 1}\left|\frac{1}{2}(\widehat{U_n}(y)-y)^2 {J_\tau}''(\theta_{2n}(y))\right| = O((n^{-D}L(n)\log\log n)^\tau) \quad \text{a.s.}$$

Hence

$$\sup_{0\le y\le 1}|J_\tau(\widehat{U_n}(y)) - J_\tau(y) + J'_\tau(y)n^{-1}V(y,n)| = O((n^{-D}L(n)\log\log n)^\tau) \quad \text{a.s.}$$

Now (2.7) with $t=1$ implies

$$\limsup_{n\to\infty}(n^{-D}L(n)\log\log n)^{-\tau/2}\sup_{0\le y\le 1}|J_\tau(\widehat{U_n}(y)) - J_\tau(y)| = \frac{2^{(\tau+1)/2}\kappa_{2\tau}}{\sqrt{\tau!(2-\tau D)(1-\tau D)}}$$

almost surely and, again by (2.7), we conclude that

$$\begin{aligned}&\limsup_{n\to\infty} n^{\tau D-1}(L(n)\log\log n)^{-\tau}\sup_{0\le y\le 1}|V(\widehat{U_n}(y),n) - V(y,n)|\\ &= \tfrac{2^{\tau+1}\kappa_{2\tau}}{\tau!(2-\tau D)(1-\tau D)} \quad \text{a.s.}\end{aligned}$$

Hence we have, with $t\in(0,1)$ fixed, as $(nt)\to\infty$,

$$\begin{aligned}&\sup_{0\le y\le 1}|V(\widehat{U_{[nt]}}(y),nt) - V(y,nt)|\\ &= \left(\tfrac{2^{\tau+1}\kappa_{2\tau}}{\tau!(2-\tau D)(1-\tau D)} + o(1)\right)(nt)^{1-\tau D}L^\tau(nt)(\log\log(nt))^\tau \quad \text{a.s.,}\end{aligned}$$

and hence, on dividing both sides by $n^{1-\tau D}L^\tau(n)(\log\log n)^\tau$ and assuming without loss of generality that the regularly varying function $n^{1-\tau D}L^\tau(n)$ of positive exponent is strictly monotone increasing, taking $\sup_{0\le t\le 1}$ on both sides, we conclude the proof of Proposition 2.3. $\square$

Proposition 2.4 *Under the assumptions of* Proposition 2.2*, as $n\to\infty$, we have*

$$\begin{aligned}&\sup_{0\le t\le 1}\sup_{0\le y\le 1}|V(\widehat{U_{[nt]}}(y),nt) - V(y-[nt]^{-1}V(y,nt),nt)|\\ &= O(n^{1-3\tau D/2}(L(n)\log\log n)^{3\tau/2}) \quad \textit{a.s.}\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\sup_{0\le t\le 1}\sup_{0\le y\le 1}|(V(\widehat{U_{[nt]}}(y),nt) - V(y,nt)) - (V(y-[nt]^{-1}V(y,nt),nt) - V(y,nt))|\\ &= O(n^{1-3\tau D/2}(L(n)\log\log n)^{3\tau/2}) \quad \textit{a.s.}\end{aligned}$$

Proof. Notice that

$$V(\widehat{U_n}(y), n) = V(y - n^{-1}V(y,n) - \Delta_n(y), n),$$

where $\Delta_n(y) = d_n n^{-1}(u_n(y,1) - d_n^{-1}V(y,n))$. By Proposition 2.2 with $t = 1$, we get

$$\sup_{0\le y\le 1} |\Delta_n(y)| = O((n^{-D}L(n)\log\log n)^{\tau}) \quad \text{a.s.}$$

Consequently, along the lines of the proof for (2.11), we obtain

$$\sup_{0\le y\le 1} \left| V(\widehat{U_n}(y), n) - V(y - n^{-1}V(y,n), n) \right| = O(n^{1-3\tau D/2}L^{3\tau/2}(n)(\log\log n)^{3\tau/2}) \quad \text{a.s.}$$

This also completes the proof of Proposition 2.4 by using a similar argument as in the end of the proof of Proposition 2.3. $\square$

Proposition 2.5 *Under the assumptions of* Proposition 2.2, *we have*

$$\begin{aligned} &\sup_{0\le t\le 1}\sup_{0\le y\le 1} |V(y - [nt]^{-1}V(y,nt), nt) - V(y,nt) + [nt]^{-1}V(y,nt)J'_\tau(y)\sum_{i=1}^{[nt]} H_\tau(\eta_i)/\tau!| \\ &= O(n^{1-3\tau D/2}(L(n)\log\log n)^{3\tau/2}) \quad a.s., \quad n\to\infty, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &\limsup_{n\to\infty} n^{\tau D-1}(L(n)\log\log n)^{-\tau} \sup_{0\le t\le 1}\sup_{0\le y\le 1} |V(y-[nt]^{-1}V(y,nt),nt) - V(y,nt)| \\ &= \limsup_{n\to\infty} n^{\tau D-1}(L(n)\log\log n)^{-\tau} \sup_{0\le t\le 1}\sup_{0\le y\le 1} |[nt]^{-1}V(y,nt)J'_\tau(y)\sum_{i=1}^{[nt]} H_\tau(\eta_i)/\tau!| \\ &= \tfrac{2^{\tau+1}\kappa_{2\tau}}{\tau!(2-\tau D)(1-\tau D)} \quad a.s. \end{aligned} \tag{2.14}$$

Proof. By (2.8) and (2.10) respectively, as $n\to\infty$, we have

$$\sup_{0\le y\le 1} n^{-1}|V(y,n)| = O((n^{-D}L(n)\log\log n)^{\tau/2}) \quad \text{a.s.}$$

and

$$\sup_{0\le y\le 1} |\widehat{U_n}(y) - y| = O((n^{-D}L(n)\log\log n)^{\tau/2}) \quad \text{a.s.}$$

Hence, along the lines of the proof of Proposition 2.3, we first obtain (2.13) and (2.14) with $t = 1$, and then a similar argument as in the end of the proof of Proposition 2.3 yields (2.13) and (2.14) as stated. $\square$

2.2 Strong approximations of sequential uniform Bahadur-Kiefer process

A direct application of Corollary 2.1 and (2.5) leads to a strong approximation for the sequential uniform Bahadur-Kiefer process $R_n^*(y,t)$.

Theorem 2.1 *Under the assumptions of* Corollary 2.1, *as $n \to \infty$, we have*

$$\begin{aligned}
&\sup_{0\le t\le 1}\sup_{0\le y\le 1}|R_n^*(y,t) - (V(y,nt) - V(\widehat{U_{[nt]}}(y),nt))| \\
&= \sup_{0\le t\le 1}\sup_{0\le y\le 1}|d_n(\alpha_n(y,t) - u_n(y,t)) - (V(y,nt) - V(\widehat{U_{[nt]}}(y),nt))| \\
&= O(n^{1-\nu p/2-\tau D/4+\varepsilon}L^{\tau/2}(n)) \quad a.s.
\end{aligned}$$

Next we reformulate Theorem 2.1 as follows.

Theorem 2.2 *Under the assumptions of* Proposition 2.2, *as $n \to \infty$, we have*

$$\begin{aligned}
&\sup_{0\le t\le 1}\sup_{0\le y\le 1}\left|[nt]R_n^*(y,t) - J_\tau(y)J_\tau'(y)\left(\sum_{i=1}^{[nt]} H_\tau(\eta_i)/\tau!\right)^2\right| \\
&= O(n^{2-\nu p/2-\tau D/4+\varepsilon}L^{\tau/2}(n)) \quad a.s.
\end{aligned}$$

Proof. Propositions 2.4-2.5 and Theorem 2.1 imply the result. $\square$

These strong approximations readily yield weak convergence and laws of the iterated logarithm for the process $R_n^*(y,t)$.

Theorem 2.3 *Under the assumptions of* Proposition 2.2, *as $n \to \infty$, we have*

$$n^{\tau D-2}L^{-\tau}(n)[nt]R_n^*(y,t) \xrightarrow{\mathcal{D}} \frac{2}{(2-\tau D)(1-\tau D)}J_\tau(y)J_\tau'(y)Y_\tau^2(t)$$

in the space $D[0,1]^2$, equipped with sup-norm, where $Y_\tau(t)$ is as in Theorem A.

Proof. From Theorem 5.6 of Taqqu (1979), as $n \to \infty$, we conclude

$$d_n^{-1} \sum_{i=1}^{[nt]} H_\tau(\eta_i)/\tau! \xrightarrow{\mathcal{D}} \sqrt{\frac{2}{(2-\tau D)(1-\tau D)}} Y_\tau(t)$$

in $D[0,1]$. Now Theorem 2.3 follows from Theorem 2.2. $\square$

In the light of Theorems 2.2-2.3 we have the following

Theorem 2.4 *Under the assumptions of* Proposition 2.2, *we have*

$$\limsup_{n\to\infty} n^{\tau D-2}(L(n)\log\log n)^{-\tau} \sup_{0\le t\le 1}\sup_{0\le y\le 1} |[nt]R_n^*(y,t)| = \frac{2^{\tau+1}\kappa_{2\tau}}{\tau!(2-\tau D)(1-\tau D)} \quad a.s. \tag{2.15}$$

as well as

$$n^{\tau D-2}L^{-\tau}(n) \sup_{0\le t\le 1}\sup_{0\le y\le 1} |[nt]R_n^*(y,t)| \xrightarrow{\mathcal{D}} \frac{2\kappa_{2\tau}}{(2-\tau D)(1-\tau D)} \sup_{0\le t\le 1} Y_\tau^2(t), \quad n\to\infty. \tag{2.16}$$

Proof. (2.15) follows from Theorem 2.2 and the law of the iterated logarithm (2.7) for $\sum_{i=1}^{[nt]} H_\tau(\eta_i)/\tau!$. As to (2.16), it results from Theorem 2.3 directly. $\square$

Denote the L_p-norm of a function f on $[0,1]^2$ by

$$\|f\|_p = \left(\int_0^1\int_0^1 |f(y,t)|^p dydt\right)^{1/p}, \quad 1\le p<\infty.$$

A straightforward L_p-version of Theorem 2.2 for the sequential uniform Bahadur-Kiefer process $R_n^*(y,t)$ results in

Theorem 2.5 *Under the assumptions of* Proposition 2.2, *we have*

$$\limsup_{n\to\infty} n^{\tau D-2}(L(n)\log\log n)^{-\tau}\|[nt]R_n^*\|_p = \frac{2^{\tau+1}\|J_\tau J_\tau'\|_p}{\tau!(2-\tau D)(1-\tau D)} \quad a.s.$$

as well as

$$n^{\tau D-2}L^{-\tau}(n)\|[nt]R_n^*\|_p \xrightarrow{\mathcal{D}} \frac{2\|J_\tau J_\tau'\|_p}{(2-\tau D)(1-\tau D)}\|Y_\tau^2\|_p, \quad n\to\infty.$$

This is in contrast with the L_p-theory of the Bahadur-Kiefer process in the i.i.d. case in Csörgő and Shi (1998, 2001) which deviates substantially from its Kiefer (1967, 1970) sup-norm theory. For a review of this matter, we refer to Csáki et al. (2002). For the sake of comparison to the latter theories, Theorems 2.1-2.5 above should be read with $t = 1$. For strong approximations in sup-norm of the sequential uniform Bahadur-Kiefer process in the i.i.d. case, we refer to Csörgő and Szyszkowicz (1998).

3 Asymptotics of the uniform Vervaat error process

In support of studying the sequential uniform Vervaat error process, we first derive the weak convergence of the sequential uniform Vervaat process $V_n(\cdot, \cdot)$. This can be easily done via Theorems 2.2-2.3.

Theorem 3.1 *Under the assumptions of* Proposition 2.2, *as* $n \to \infty$, *we have*

$$V_n(s,t) \xrightarrow{\mathcal{D}} \frac{2}{(2-\tau D)(1-\tau D)} J_\tau^2(s) Y_\tau^2(t)$$

in the space $D[0,1]^2$, *equipped with sup-norm, where* $Y_\tau(t)$ *is as in* Theorem A.

Proof. Theorem 2.3 and integration by parts yield

$$\begin{aligned} V_n(s,t) = 2d_n^{-2}[nt] \int_0^s R_n^*(y,t)dy \quad &\xrightarrow{\mathcal{D}} \tfrac{4}{(2-\tau D)(1-\tau D)} \left(\int_0^s J_\tau(y) J_\tau'(y) dy \right) Y_\tau^2(t) \\ &= \tfrac{2}{(2-\tau D)(1-\tau D)} J_\tau^2(s) Y_\tau^2(t). \qquad \square \end{aligned}$$

Theorem 3.1 and Corollary A imply that the sequential uniform Vervaat process $V_n(s,t)$ and the process $\alpha_n^2(s,t)$ have the same weak limiting process. Thus, just like in the i.i.d. case, it makes sense to consider the deviation of the two processes, i.e., the sequential uniform Vervaat error process Q_n as in (1.10), namely

$$Q_n(s,t) = 2d_n^{-2}[nt] \int_0^s R_n^*(y,t)dy - \alpha_n^2(s,t), \quad 0 \le s \le 1, 0 \le t \le 1.$$

Unlike in the i.i.d. case (cf. Csáki et al. (2002)), we shall see that $Q_n(s,1)$, as well as its sequential version $Q_n(s,t)$, do converge weakly, and in particular to a random process which is a multiplication of a non-random function with the cube of random process $Y_\tau(t)$ defined in Theorem A.

Proposition 3.1 *Under the assumptions of* Proposition 2.2, *we have*

$$\sup_{0\le t\le 1}\sup_{0\le s\le 1}[nt]|Q_n(s,t)-Z_n(s,t)| = O(n^{1-\nu p/2+\tau D/4+\varepsilon}(\log\log n)^{\tau/2}), \quad a.s.$$

where $\{Z_n(s,t), 0\le s,t\le 1, n=1,2,\cdots\}$ *is defined by*

$$Z_n(s,t) = 2d_n^{-2}V(s,nt)\int_0^1\Big(V(s-w[nt]^{-1}V(s,nt),nt)-V(s,nt)\Big)\,dw. \tag{3.1}$$

Proof. We proceed *à la* the lines of the proofs of Lemmas 3.1 and 3.2 of Csáki et al. (2002). Let

$$A_n(s,t) = 2d_n^{-1}[nt]\int_{\widehat{U_{[nt]}}(s)}^{s}(\alpha_n(y,t)-\alpha_n(s,t))\,dy, \quad 0\le s,t\le 1, n=1,2,\cdots. \tag{3.2}$$

It follows from Lemma 3.1 of Csáki et al. (2002) that

$$Q_n(s,t) = A_n(s,t) - d_n^{-2}(R_n^*(s,t))^2.$$

Now (2.15) with $t=1$ yields that, when $n\to\infty$,

$$\sup_{0\le s\le 1} n|Q_n(s,1)-A_n(s,1)| = O(n^{1-\tau D}L^\tau(n)(\log\log n)^{2\tau}) \quad \text{a.s.}$$

By a similar fashion as in the end of the proof of Proposition 2.1, as $n\to\infty$, we get

$$\sup_{0\le t\le 1}\sup_{0\le s\le 1}[nt]|Q_n(s,t)-A_n(s,t)| = O(n^{1-\tau D}L^\tau(n)(\log\log n)^{2\tau}) \quad \text{a.s.}$$

Hence, it suffices to show that, as $n\to\infty$,

$$\sup_{0\le t\le 1}\sup_{0\le s\le 1}[nt]|A_n(s,t)-Z_n(s,t)| = O(n^{1-\nu p/2+\tau D/4+\varepsilon}(\log\log n)^{\tau/2}) \quad \text{a.s.}$$

Changing variable $y = s - w(s - \widehat{U_{[nt]}}(s)) = s - w[nt]^{-1}d_n u_n(s,t)$ in (3.2), we get

$$A_n(s,t) = 2u_n(s,t)\int_0^1 \Big(\alpha_n(s - w[nt]^{-1}d_n u_n(s,t)) - \alpha_n(s,t)\Big)\,dw.$$

Corollary 2.1 and (2.9), as $n \to \infty$, yield

$$\begin{aligned} A_n(s,t) = \;& 2d_n^{-1}u_n(s,t)\int_0^1 \Big(V(s - w[nt]^{-1}d_n u_n(s,t), nt) - V(s,nt)\Big)\,dw \\ & + O(n^{-\nu p/2 + \tau D/4 + \varepsilon}(\log\log n)^{\tau/2}) \quad \text{a.s.,} \end{aligned} \tag{3.3}$$

uniformly in $s,t \in [0,1]$. For all $0 \le w \le 1$, according to Proposition 2.4, as $n \to \infty$, we have uniformly in $s,t \in [0,1]$

$$\begin{aligned} V(s - w[nt]^{-1}d_n u_n(s,t), nt) = \;& V(s - w[nt]^{-1}V(s,nt), nt) \\ & + O(n^{1-3\tau D/2}(L(n)\log\log n)^{3\tau/2}) \quad \text{a.s.} \end{aligned}$$

Inserting this into (3.3) and applying (2.9) again, we obtain uniformly in $s,t \in [0,1]$

$$\begin{aligned} A_n(s,t) \;&= 2d_n^{-1}u_n(s,t)\int_0^1 \Big(V(s - w[nt]^{-1}V(s,nt), nt) - V(s,nt)\Big)\,dw \\ & + O(n^{-\nu p/2 + \tau D/4 + \varepsilon}(\log\log n)^{\tau/2}) + O(n^{-\tau D}L^{\tau}(n)(\log\log n)^{2\tau}) \quad \text{a.s.} \end{aligned}$$

Consequently, as $n \to \infty$, uniformly in $s,t \in [0,1]$,

$$\begin{aligned} A_n(s,t) \;&= 2d_n^{-1}u_n(s,t)\int_0^1 \Big(V(s - w[nt]^{-1}V(s,nt), nt) - V(s,nt)\Big)\,dw \\ & + O(n^{-\nu p/2 + \tau D/4 + \varepsilon}(\log\log n)^{\tau/2}) \quad \text{a.s.} \end{aligned} \tag{3.4}$$

Now, from Proposition 2.2, as $n \to \infty$,

$$2d_n^{-1}u_n(s,t) = 2d_n^{-2}V(s,nt) + O(n^{-1}(\log\log n)^{\tau}) \quad \text{a.s.} \tag{3.5}$$

uniformly in $0 \le s,t \le 1$. On the other hand, applying (2.14) to the integrand in (3.4), we arrive at

$$\int_0^1 \Big(V(s - w[nt]^{-1}V(s,nt), nt) - V(s,nt)\Big)\,dw = O(n^{1-\tau D}(L(n)\log\log n)^{\tau}) \quad \text{a.s.}$$

uniformly in $0 \le s,t \le 1$. Inserting this and (3.5) into (3.4) yields that, as $n \to \infty$,

$$[nt]|A_n(s,t) - Z_n(s,t)| = O(n^{1-\nu p/2 + \tau D/4 + \varepsilon}(\log\log n)^{\tau/2}) \quad \text{a.s.}$$

uniformly in $0 \leq s, t \leq 1$. This concludes the proof of Proposition 3.1. $\Box$

Due to Proposition 2.5, we present the following conclusion.

Proposition 3.2 *Under the assumptions of* Proposition 2.2, *we have*

$$\sup_{0\leq t\leq 1}\sup_{0\leq s\leq 1}|Z_n(s,t)+2d_n^{-2}[nt]^{-1}(V(s,nt))^2 J_\tau'(s)\sum_{i=1}^{[nt]}H_\tau(\eta_i)/\tau!|$$

$$= O(n^{-\tau D}L^\tau(n)(\log\log n)^{2\tau}) \quad a.s.$$

Proof. By (3.1) and Proposition 2.5, we obtain

$$\begin{aligned}
&\sup_{0\leq t\leq 1}\sup_{0\leq s\leq 1}|Z_n(s,t)+2d_n^{-2}[nt]^{-1}(V(s,nt))^2 J_\tau'(s)\sum_{i=1}^{[nt]}H_\tau(\eta_i)/\tau!|\\
&=\sup_{0\leq t\leq 1}\sup_{0\leq s\leq 1}|2d_n^{-2}V(s,nt)\cdot\\
&\int_0^1\left\{V(s-w[nt]^{-1}V(s,nt),nt)-V(s,nt)+w[nt]^{-1}V(s,nt)J_\tau'(s)\sum_{i=1}^{[nt]}H_\tau(\eta_i)/\tau!\right\}dw|\\
&\leq \sup_{0\leq t\leq 1}\sup_{0\leq s\leq 1}|2d_n^{-2}V(s,nt)|\cdot\\
&\sup_{0\leq t\leq 1}\sup_{0\leq s,w\leq 1}|V(s-w[nt]^{-1}V(s,nt),nt)-V(s,nt)+w[nt]^{-1}V(s,nt)J_\tau'(s)\sum_{i=1}^{[nt]}H_\tau(\eta_i)/\tau!|\\
&=O(n^{-\tau D}L^\tau(n)(\log\log n)^{2\tau}) \quad \text{a.s.}
\end{aligned}$$

This completes the proof. $\Box$

The main conclusions of this section are as follows.

Theorem 3.2 *Under the assumptions of* Proposition 2.2, *as $n\to\infty$, we have*

$$n^{\tau D/2-1}L^{-\tau/2}(n)[nt]Q_n(s,t)\xrightarrow{\mathcal{D}}2^{5/2}\left((2-\tau D)(1-\tau D)\right)^{-3/2}J_\tau^2(s)J_\tau'(s)Y_\tau^3(t)$$

in the space $D[0,1]^2$, equipped with sup-norm, where $Y_\tau(t)$ is as in Theorem A.

Proof. It follows from Theorem 5.6 of Taqqu (1979) and Propositions 3.1-3.2, $\Box$

As a consequence of Propositions 3.1-3.2 and Theorem 3.2 we have the following results.

Theorem 3.3 *Under the conditions of* Proposition 2.2, *we have*

$$\limsup_{n\to\infty} n^{\tau D/2-1}L^{-\tau/2}(n)(\log\log n)^{-3\tau/2}\sup_{0\le t\le 1}\sup_{0\le s\le 1}|[nt]Q_n(s,t)|$$
$$= 2^{(3\tau+5)/2}\kappa_{3\tau}\left(\tau!(2-\tau D)(1-\tau D)\right)^{-3/2} \quad a.s.,$$

and, as $n\to\infty$,

$$n^{\tau D/2-1}L^{-\tau/2}(n)\quad \sup_{0\le t\le 1}\sup_{0\le s\le 1}|[nt]Q_n(s,t)|$$
$$\xrightarrow{\mathcal{D}} 2^{5/2}\kappa_{3\tau}\left((2-\tau D)(1-\tau D)\right)^{-3/2}\sup_{0\le t\le 1}|Y_\tau^3(t)|.$$

Moreover,

$$\limsup_{n\to\infty} n^{\tau D/2-1}L^{-\tau/2}(n)(\log\log n)^{-3\tau/2}\|[nt]Q_n\|_p$$
$$= 2^{(3\tau+5)/2}\|J_\tau^2 J_\tau'\|_p\left(\tau!(2-\tau D)(1-\tau D)\right)^{-3/2} \quad a.s.,$$

and, as $n\to\infty$,

$$n^{\tau D/2-1}L^{-\tau/2}(n)\|[nt]Q_n\|_p \xrightarrow{\mathcal{D}} 2^{5/2}\|J_\tau^2 J_\tau'\|_p\left((2-\tau D)(1-\tau D)\right)^{-3/2}\|Y_\tau^3\|_p,$$

where, in both cases, Y_τ *is as in* Theorem A.

Reading Theorems 3.2 and 3.3 with $t=1$, they should be compared to Theorem 2.1, and Corollaries 2.1 and 2.2, of Csáki et al. (2002) in the i.i.d. case.

4 Sequential general Bahadur-Kiefer processes, Strong approximations

In this section we shall study the sequential general Bahadur-Kiefer process $R_n(y,t)$ in terms of the sequential uniform Bahadur-Kiefer process $R_n^*(y,t)$.

The following Csáki-type law of the iterated logarithm (cf. Csáki (1977)) for the sequential uniform quantile process plays a crucial role in comparing the two processes $\rho_n(y,t)$ and $u_n(y,t)$.

Proposition 4.1 *Assume that the assumptions of* Proposition 2.2 *hold, then as* $n\to\infty$, *we have*

$$\sup_{\delta_n\le y\le 1-\delta_n}|u_n(y,1)|^2/J_\tau^2(y) = O((\log\log n)^\tau) \quad a.s.,$$

where $\delta_n = (n^{-D}L(n)\log\log n)^{\tau}$.

Proof. Note that

$$\begin{aligned}
&\sup_{\delta_n \le y \le 1/2} \left| u_n^2(y,1) - |d_n^{-1}V(y,n)|^2 \right| /y \\
&\le \sup_{0\le y\le 1} |u_n(y,1) - d_n^{-1}V(y,n)|^2 \cdot \delta_n^{-1} + 2 \sup_{\delta_n \le y \le 1/2} |d_n^{-1}V(y,n)|/y^{1/2} \\
&\cdot \sup_{0\le y\le 1} |u_n(y,1) - d_n^{-1}V(y,n)| \cdot \delta_n^{-1/2}.
\end{aligned}$$

Assumption (A) and simple calculations yield

$$\sup_{\delta_n \le y \le 1/2} |J_\tau(y)|/y^{1/2} = O(1), \quad \text{and} \quad \sup_{1/2 \le y \le 1-\delta_n} |J_\tau(y)|/(1-y)^{1/2} = O(1) \tag{4.1}$$

for large enough n. Consequently, (2.4), (2.7) and (4.1) imply that, as $n \to \infty$,

$$\sup_{\delta_n \le y \le 1/2} \left| u_n^2(y,1) - |d_n^{-1}V(y,n)|^2 \right| /y = O((\log\log n)^{\tau}) \quad \text{a.s.}$$

Similarly, as $n \to \infty$, we get

$$\sup_{1/2 \le y \le 1-\delta_n} \left| u_n^2(y,1) - |d_n^{-1}V(y,n)|^2 \right| /(1-y) = O((\log\log n)^{\tau}) \quad \text{a.s.}$$

This, in turn, results in

$$\sup_{\delta \le y \le 1-\delta_n} \frac{|u_n^2(y,1) - |d_n^{-1}V(y,n)|^2|}{y(1-y)} = O((\log\log n)^{\tau}) \quad \text{a.s.} \tag{4.2}$$

On the other hand, by (2.8) and (4.1), we know that as $n \to \infty$

$$\sup_{\delta_n \le y \le 1-\delta_n} |d_n^{-1}V(y,n)|^2/(y(1-y)) = O((\log\log n)^{\tau}) \quad \text{a.s.}$$

Thus, via (4.2), as $n \to \infty$ we arrive at

$$\sup_{\delta_n \le y \le 1-\delta_n} |u_n(y,1)|^2/(y(1-y)) = O((\log\log n)^{\tau}) \quad \text{a.s.} \tag{4.3}$$

Now (4.1) and (4.3) yield the result of Proposition 4.1. □

In the light of Proposition 4.1, and Lemma 1 of Csörgő and Révész (1978) (cf. Lemma 4.5.2 in Csörgő and Révész (1981)), it is natural to introduce the following conditions:

(i) F is twice differentiable on (a,b), where

$$a = \sup\{x : F(x) = 0\}, \quad b = \inf\{x : F(x) = 1\}, \quad -\infty \leq a < b \leq +\infty;$$

(ii) $F'(x) = f(x) > 0$ on (a,b);

(iii) for some $0 < \gamma < 1 + (\tau D)/(2 - 2\tau D)$, where $\nu = \min(D, 1 - \tau D)/2$, we have

$$\sup_{a<x<b} J_\tau^2(F(x)) \frac{|f'(x)|}{f^2(x)} = \sup_{0<y<1} J_\tau^2(y) \frac{|f'(Q(y))|}{f^2(Q(y))} \leq \gamma;$$

(iv) $A := \overline{\lim}_{x \downarrow a} f(x) < \infty$, $B := \overline{\lim}_{x \uparrow b} f(x) < \infty$;

(v) $\min(A,B) > 0$, or

(v') if $A = 0$ (resp. $B = 0$), then f is non-decreasing (resp. non-increasing) on an interval to the right of a (resp. to the left of b).

Remark 4.1 Initially similar conditions were introduced in Csörgő and Révész (1981), which were then further studied and utilized in Csörgő (1983), Csörgő et al. (1985), Csörgő and Horváth (1993), Csörgő and Szyszkowicz (1998), Csörgő and Shi (2001), and Csörgő and Zitikis (2002). Our condition (iii) is slightly stronger than the corresponding one in the i.i.d. case in the above mentioned works. It can actually be replaced by the similarly stronger condition

(iii)* for some $0 < \gamma < 1 + (\tau D)/(2 - 2\tau D)$, we have

$$\sup_{a<x<b} F(x)(1 - F(x)) \frac{|f'(x)|}{f^2(x)} = \sup_{0<y<1} y(1-y) \frac{|f'(Q(y))|}{f^2(Q(y))} \leq \gamma,$$

whenever (iii) is assumed in the sequel below.

With (iii)* in mind now, we mention examples of distributions which satisfy our just mentioned condition (iii)*, which is easier to calculate with than with (iii). For example,

if $F(x) = 1 - e^{-x}$, $x \geq 0$, then $f(Q(y)) = 1 - y$, $f'(Q(y)) = -1$. Therefore γ of (iii) is equal to 1;

if $F(x) = x$, $0 < x < 1$, then $f(Q(y)) = 1$, $f'(Q(y)) = 0$. Then γ of (iii)* can be 1/2;

if $F(x) = \Phi(x)$, $-\infty < x < \infty$, then $f(Q(y)) = \phi(\Phi^{-1}(y))$, $f'(Q(y)) = -\Phi^{-1}(y)\phi(\Phi^{-1}(y))$. Elementary calculations yield that

$$\sup_{0<y<1/2} y(1-y)|-\Phi^{-1}(y)|/\phi(\Phi^{-1}(y)) \le 1+\epsilon$$

and

$$\sup_{1/2\le y<1} y(1-y)|-\Phi^{-1}(y)|/\phi(\Phi^{-1}(y)) \le 1+\epsilon,$$

where ϵ $(< (\tau D)/(2-2\tau D))$ is a small positive constant. Hence γ of (iii)* can be selected from the interval $(1, 1+(\tau D)/(2-2\tau D))$.

The following Proposition studies the sup-norm distance between $\rho_n(y,t)$ and $u_n(y,t)$.

Proposition 4.2 *Assume the conditions* (i)–(iii) *on F and the assumptions of* Proposition 2.2. *Then, as $n \to \infty$, we have*

$$\sup_{0\le t\le 1}\sup_{\delta_n\le y\le 1-\delta_n} |\rho_n(y,t) - d_n^{-1}V(y,nt)| = O(n^{-\tau D/2}L^{\tau/2}(n)(\log\log n)^{\tau}) \quad a.s., \tag{4.4}$$

where $\delta_n = (n^{-D}L(n)\log\log n)^{\tau}$. If, in addition to (i)–(iii), *we also assume* (iv) *and* (v) (*or* (v')). *Then*

$$\sup_{0\le t\le 1}\sup_{0\le y\le 1} |\rho_n(y,t) - d_n^{-1}V(y,nt)| =$$
$$\left\{\begin{array}{ll} O\left(n^{-\tau D/2}L^{\tau/2}(n)(\log\log n)^{\tau+1}\right), & 0<\gamma\le 1, \\ O\left(n^{(1-\tau D)\gamma+\tau D/2-1}L^{\tau\gamma-\tau/2}(n)(\log n)^{(1+C)(\gamma-1)}\right), & 1<\gamma<1+\frac{\tau D}{2(1-\tau D)}, \end{array}\right. \quad a.s. \tag{4.5}$$

where $C > 0$ is arbitrary.

Proof. Observe that a two-term Taylor expansion gives

$$\begin{aligned} \rho_n(y,1) &= d_n^{-1}nf(Q(y))(Q(y)-\widehat{Q_n}(y)) = d_n^{-1}nf(Q(y))(Q(y)-Q(\widehat{U_n}(y))) \\ &= u_n(y,1) - \tfrac{d_n}{2n}u_n^2(y)\tfrac{f'(Q(\theta_{3n}(y)))}{f^3(Q(\theta_{3n}(y)))}f(Q(y)), \end{aligned} \tag{4.6}$$

where $|y-\theta_{3n}(y)| \le |y-\widehat{U_n}(y)|$.

By (4.1), arguing as in the proof of Theorem 4.5.6 of Csörgő and Révész (1981), we arrive at

$$\sup_{0<\theta_{3n}(y)<1} J_\tau^2(\theta_{3n}(y))\frac{|f'(Q(\theta_{3n}(y)))|}{f^2(Q(\theta_{3n}(y)))} \le \gamma,$$

and

$$\sup_{\delta_n \le y \le 1-\delta_n} \frac{f(Q(y))}{f(Q(\theta_{3n}(y)))} \le \sup_{\delta_n \le y \le 1-\delta_n} \left[\frac{\theta_{3n}(y)(1-y)}{y(1-\theta_{3n}(y))} + \frac{y(1-\theta_{3n}(y))}{\theta_{3n}(y)(1-y)} \right]^{\gamma} < \infty.$$

These, together with Proposition 4.1 and (4.6), yield

$$\sup_{\delta_n \le y \le 1-\delta_n} |\rho_n(y,1) - u_n(y,1)| = O(n^{-\tau D/2} L^{\tau/2}(n)(\log\log n)^{\tau}) \quad \text{a.s.}$$

Hence, arguing as in the end of the proof of Proposition 2.1, we conclude

$$\sup_{0 \le t \le 1} \sup_{\delta_n \le y \le 1-\delta_n} |\rho_n(y,t) - u_n(y,t)| = O(n^{-\tau D/2} L^{\tau/2}(n)(\log\log n)^{\tau}) \quad \text{a.s.} \tag{4.7}$$

Now (2.4) and (4.7) together imply (4.4).

Next, assuming now (iv) and (v), consider the one-term Taylor expansion as in (1.6),

$$\rho_n(y,t) = u_n(y,t) \frac{f(Q(y))}{f(Q(\theta_n(y,t)))}.$$

It follows from Assumption (A) in combination with (2.4) and (2.7) that

$$\sup_{0 \le t \le 1} \sup_{0 \le y \le \delta_n} |u_n(y,t)| = O(n^{-\tau D} L^{\tau}(n)(\log\log n)^{3\tau/2}) \quad \text{a.s.} \tag{4.8}$$

and

$$\sup_{0 \le t \le 1} \sup_{1-\delta_n \le y \le 1} |u_n(y,t)| = O(n^{-\tau D} L^{\tau}(n)(\log\log n)^{3\tau/2}) \quad \text{a.s.}$$

Hence we have

$$\sup_{0 \le t \le 1} \sup_{0 \le y \le \delta_n} |\rho_n(y,t) - u_n(y,t)| = O(n^{-\tau D/2} L^{\tau/2}(n)(\log\log n)^{\tau}) \quad \text{a.s.} \tag{4.9}$$

and

$$\sup_{0 \le t \le 1} \sup_{1-\delta_n \le y \le 1} |\rho_n(y,t) - u_n(y,t)| = O(n^{-\tau D/2} L^{\tau/2}(n)(\log\log n)^{\tau}) \quad \text{a.s.} \tag{4.10}$$

Using (2.4), (4.7) and (4.9)-(4.10), we get

$$\sup_{0 \le t \le 1} \sup_{0 \le y \le 1} |\rho_n(y,t) - d_n^{-1} V(y,nt)| = O(n^{-\tau D/2} L^{\tau/2}(n)(\log\log n)^{\tau}) \quad \text{a.s.} \tag{4.11}$$

Finally, we assume (iv) and (v'). In order to prove (4.5), it again suffices to show that $\sup_{0\le t\le 1}\sup_{0\le y\le \delta_n}|\rho_n(y,t)-u_n(y,t)|$ and $\sup_{0\le t\le 1}\sup_{1-\delta_n\le y\le 1}|\rho_n(y,t)-u_n(y,t)|$ converge to zero a.s. under assumption (iv) and (v'). We demonstrate this only for the first one of these, since for the second one a similar argument holds.

Along similar lines to the proof of Theorem 4.5.6 in Csörgő and Révész (1981), we conclude

$$|\rho_n(y,1)| \le |u_n(y,1)|, \quad \text{if} \quad \widehat{U_n}(y) \ge y,$$

and if $\widehat{U_n}(y) < y$, then

$$|\rho_n(y,1)| \le \begin{cases} O(n^{\tau D/2}L^{-\tau/2}(n)\delta_n), & 0<\gamma<1, \\ O(n^{\tau D/2}L^{-\tau/2}(n)\delta_n \log\log n), & \gamma = 1, \\ O(n^{\tau D/2}L^{-\tau/2}(n)\delta_n^{\gamma} n^{\gamma-1}(\log n)^{(1+C)(\gamma-1)}), & 1<\gamma<1+\frac{\tau D}{2(1-\tau D)}, \end{cases} \quad \text{a.s.}$$

where $C>0$ is arbitrary. Note that $-\tau D/2 < (1-\tau D)\gamma + \tau D/2 - 1 < 0$ if $1<\gamma<1+(\tau D)/(2-2\tau D)$. Hence, with the help of (4.8), we obtain

$$\begin{aligned}&\sup_{0\le y\le \delta_n}|\rho_n(y,1)-u_n(y,1)| = \\ &\begin{cases} O(n^{-\tau D/2}L^{\tau/2}(n)(\log\log n)^{\tau+1}), & 0<\gamma\le 1, \\ O(n^{(1-\tau D)\gamma+\tau D/2-1}L^{\tau\gamma-\tau/2}(n)(\log n)^{(1+C)(\gamma-1)}), & 1<\gamma<1+\frac{\tau D}{2(1-\tau D)} \end{cases} \quad \text{a.s.}\end{aligned}$$

This, combined with Proposition 2.2 and (4.11), completes the proof of Proposition 4.2. $\square$

Note that

$$\begin{aligned}&\{R_n(y,t)-R_n^*(y,t), 0\le y,t\le 1, n=1,2\cdots\} \\ &=\{-d_n(\rho_n(y,t)-u_n(y,t)), 0\le y,t\le 1, n=1,2\cdots\}\end{aligned} \tag{4.12}$$

The relationship (4.12) clearly indicates that the results we have summarized and proved in Theorems 2.3-2.5 for $R_n^*(y,t)$ can be immediately restated for the sequential general Bahadur-Kiefer process $R_n(y,t)$ via the strong invariance principle of Proposition 4.2. So we spell out and summarize these results for $R_n(y,t)$ without proof.

Theorem 4.1 *Assume the conditions* (i)–(iii) *on F and the assumptions of* Proposition 2.2, *then as $n\to\infty$, we have*

$$n^{\tau D-2}L^{-\tau}(n)[nt]R_n(y,t)I\{\delta_n\le y\le 1-\delta_n\} \xrightarrow{\mathcal{D}} \frac{2}{(2-\tau D)(1-\tau D)}J_\tau(y)J_\tau'(y)Y_\tau^2(t)$$

in the space $D[0,1]^2$*, equipped with the sup-norm, where* $\delta_n = (n^{-D}L(n)\log\log n)^{\tau}$. *Moreover,*

$$\limsup_{n\to\infty} n^{\tau D-2}(L(n)\log\log n)^{-\tau} \sup_{0\le t\le 1}\sup_{\delta_n\le y\le 1-\delta_n} |[nt]R_n(y,t)| = \frac{2^{\tau+1}\kappa_{2\tau}}{\tau!(2-\tau D)(1-\tau D)} \quad a.s.,$$

and, as $n\to\infty$*,*

$$n^{\tau D-2}L^{-\tau}(n) \sup_{0\le t\le 1}\sup_{\delta_n\le y\le 1-\delta_n} |[nt]R_n(y,t)| \xrightarrow{\mathcal{D}} \frac{2\kappa_{2\tau}}{(2-\tau D)(1-\tau D)} \sup_{0\le t\le 1} Y_\tau^2(t).$$

Theorem 4.2 *In addition to the conditions in* Theorem 4.1, *we assume* (iv) *and* (v)(*or* (v')), *then as* $n\to\infty$*, we have*

$$n^{\tau D-2}L^{-\tau}(n)[nt]R_n(y,t) \xrightarrow{\mathcal{D}} \frac{2}{(2-\tau D)(1-\tau D)} J_\tau(y)J'_\tau(y)Y_\tau^2(t)$$

in the space $D[0,1]^2$*, equipped with sup-norm, as well as*

$$\limsup_{n\to\infty} n^{\tau D-2}(L(n)\log\log n)^{-\tau} \sup_{0\le t\le 1}\sup_{0\le y\le 1} |[nt]R_n(y,t)| = \frac{2^{\tau+1}\kappa_{2\tau}}{\tau!(2-\tau D)(1-\tau D)} \quad a.s.$$

and

$$n^{\tau D-2}L^{-\tau}(n) \sup_{0\le t\le 1}\sup_{0\le y\le 1} |[nt]R_n(y,t)| \xrightarrow{\mathcal{D}} \frac{2\kappa_{2\tau}}{(2-\tau D)(1-\tau D)} \sup_{0\le t\le 1} Y_\tau^2(t), n\to\infty.$$

Moreover,

$$\limsup_{n\to\infty} n^{\tau D-2}(L(n)\log\log n)^{-\tau}\|[nt]R_n\|_p = \frac{2^{\tau+1}\|J_\tau J'_\tau\|_p}{\tau!(2-\tau D)(1-\tau D)} \quad a.s.$$

and, as $n\to\infty$*,*

$$n^{\tau D-2}L^{-\tau}(n)\|[nt]R_n\|_p \xrightarrow{\mathcal{D}} \frac{2\|J_\tau J'_\tau\|_p}{(2-\tau D)(1-\tau D)}\|Y_\tau^2\|_p.$$

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