Censoring, Factorizations, and Spectral Analysis for Transition Matrices with Block-Repeating Entries

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Abstract

In this paper, we use the Markov chain censoring technique to study infinite state Markov chains whose transition matrices possess block-repeating entries. We demonstrate that a number of important probabilistic measures are invariant under censoring. Informally speaking, these measures involve first passage times or expected numbers of visits to certain levels where other levels are taboo; they are closely related to the so-called fundamental matrix of the Markov chain which is also studied here. Factorization theorems for the characteristic equation of the blocks of the transition matrix are obtained. Necessary and sufficient conditions are derived for such a Markov chain to be positive recurrent, null recurrent, or transient based either on spectral analysis, or on a property of the fundamental matrix. Explicit expressions are obtained for key probabilistic measures, including the stationary probability vector and the fundamental matrix, which could be potentially used to develop various recursive algorithms for computing these measures.

Keywords: block-Toeplitz transition matrices, factorization of characteristic functions, spectral analysis, fundamental matrix, conditions of recurrence and transience.

1 Introduction

Infinite state Markov chains with block-structured transition matrices constitute a very rich class of stochastic processes, finding applications in many areas, including telecommunications, inventory modelling, and queueing systems, for example. Markov chains of $GI/M/1$ and $M/G/1$ type are two important special cases which are now very well understood (for example, Neuts, 1980, 1989). The significance of the rate matrix R for the $GI/M/1$ type case and the matrix G of the fundamental period for the $M/G/1$ type case has been well documented, and applications of the associated matrix-analytic method are ubiquitous in the literature.

Extending matrix-analytic methods to more general block-structured Markov chains has been taken as a challenge by several researchers. Gail, Hantler and Taylor (1997) studied non-skip-free GI/M/1 and M/G/1 type Markov chains, obtaining a very useful factorization and exhibiting the special structure of the R and G matrices for these cases. Grassmann and Heyman (1990, 1993) studied general block-structured transition matrices with the aid of two sequences of matrices which generalize the R and G matrices. We refer to these two sequences as the R and G-measures, respectively.

In this paper, we wish to primarily focus on transition matrices having a block-Toeplitz or block-repeating structure. Using the concept of the censored Markov chain which we review in Section 2, we propose to elucidate the properties of the Markov chains generated by such transition matrices. Specifically, let $\{Z_t = (X_t, Y_t); t = 0, 1, 2, \ldots\}$ be the Markov chain, whose transition matrix P possesses a block structure of the form

$$P = \begin{bmatrix} D_0 & D_1 & D_2 & D_3 & \cdots & \cdots \\ D_{-1} & C_0 & C_1 & C_2 & \cdots & \cdots \\ D_{-2} & C_{-1} & C_0 & C_1 & \cdots & \cdots \\ D_{-3} & C_{-2} & C_{-1} & C_0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{1}$$

where all C_i for $i = 0, \pm 1, \pm 2, \ldots$ are matrices of size $m \times m$, D_0 is a matrix of size $m_0 \times m_0$ and the sizes of all D_i for $i = \pm 1, \pm 2, \ldots$ are determined accordingly.

We are motivated to look at transition matrices having the block-repeating form (1) since they constitute a large class, finding numerous applications. In addition, they are a natural generalization of both $M/G/1$ and $GI/M/1$ type Markov chains, offering a unifying approach to the study of these very important special cases. In particular, the R and G-measures of Grassmann and Heyman play a crucial role here. We will show that these measures are as important in the study of Markov chains with block-repeating structure as the R and G matrices are in the study of $GI/M/1$ and $M/G/1$ type Markov chains. Two other sequences of matrices, carrying probabilistic interpretations and referred to as A and B-measures, are also very useful in the block-repeating context as shown in Zhao, Li and Braun (1998). One appealing and useful property of all four sets of measures is their invariance under censoring; this will be elaborated in Section 3.

In Section 4, we will demonstrate that the above measures also have a bearing on the study of a particular matrix, called the fundamental matrix, upon which the censoring technique hinges. Some basic characterization results of the Markov chain based on the fundamental matrix will be provided.

The stage will then be set to provide, in Section 5, a factorization of the characteristic function of (1) into a product of characteristic functions for the R and G-measures. This is the content of Theorem 14. For matrices of $GI/M/1$ type and $M/G/1$ type, we will demonstrate that the factorization obtained by Gail, Hantler and Taylor (1997) is equivalent to the factorization obtained here. Characterization results for Markov chains with block-repeating transition matrices will also be provided in terms of properties of the four probabilistic measures as well as the generalized traffic intensity. The factorization theorem can also be used to compute the R and G-measures. Other key probabilistic measures, such as the fundamental matrix and the stationary probability vector, can then be efficiently determined.

Spectral analysis of the R and G matrices has been shown to be very useful in dealing with the $GI/M/1$ and $M/G/1$ paradigms (for example, Gail, Hantler and Taylor, 1996, 1997). We will show, in Section 6, and using the factorization obtained in the preceding section, that spectral analysis of the R and G-measures is also a key to characterize the Markov chains with block-repeating structure. In particular, Theorem 23 gives a general characterization.

We close this section with some notational and technical details. The transition matrix P in (1) can be either stochastic or strictly substochastic. By a strictly substochastic matrix we mean that every row sum of the transition matrix is less than or equal to one and there exists at least one row sum which is strictly less than one. The only extra condition imposed is irreducibility, though this condition may not be essential for all of the results presented in this paper. Corresponding results for continuous time Markov chains can be obtained in parallel.

2 Review of Censoring and the R and G-Measures

The censoring technique (for example, Kemeny, Snell and Knapp, 1976, Grassmann and Heyman, 1990, Zhao and Liu, 1996, or Zhao, Li and Braun, 1998) has been used in the literature in studying various aspects of Markov chains. It should be noted that stochastic complementation (for example, Meyer, 1989) and censoring are synonymous.

Censoring can be applied to an arbitrary stochastic process. However, we find that it is most effectively exploited in the context of Markov chains.

Consider a discrete-time irreducible Markov chain $\{X_n; n = 1, 2, \ldots\}$ with state space S. Let E be a non-empty subset of S. Suppose that the successive visits of X_n to E take place at time epochs $0 < n_1 < n_2 < \cdots$. Then the process $\{X_t^E = X_{n_t}; t = 1, 2, \ldots\}$ is defined as the censored process with censoring set E. Alternatively, the nth transition of the censored process is the nth time for the Markov chain to visit a state in E. From this definition, it follows that sample paths of the censored process are the paths of the original Markov chain whose transitions in the complementary set, E^c, have been deleted. Using the strong Markov property, it can be proved that the censored process is also a Markov chain, called the censored Markov chain. This new process has also been variously called the restricted, watched or embedded Markov chain.

If P is the transition matrix of the original Markov chain $\{X_n; n = 1, 2, \ldots\}$, we can partition P according to subsets E and E^c:

$$P = \begin{array}{c} \\ E \\ E^c \end{array} \begin{array}{c} \begin{array}{cc} E & E^c \end{array} \\ \left[\begin{array}{cc} T & U \\ V & Q \end{array}\right] \end{array}. \tag{2}$$

The censored transition matrix, P^E, of the censored Markov chain is then given by

$$P^E = T + U\widehat{Q}V \tag{3}$$

with $\widehat{Q} = \sum_{k=0}^{\infty} Q^k$. The matrix $\widehat{Q}$ is called the fundamental matrix of Q.

Since our later results are intimately related to the behaviour of the term $U\widehat{Q}V$ in (3), we pause to carefully review some probabilistic interpretations. In the following, $C_{i,j}$ stands for the (i,j)th entry in a matrix C, and the process referred to is the original Markov chain.

1. $(\widehat{Q})_{i,j}$ is the expected number of visits to state $j \in E^c$ before entering E given that the process started in state $i \in E^c$.

2. $(U\widehat{Q})_{i,j}$ is the expected number of visits to state $j \in E^c$ before returning to E given that the process started in state $i \in E$.

3. $(\widehat{Q}V)_{i,j}$ is the probability that the process enters E and upon entering E the first state visited is $j \in E$, given that the process started in state $i \in E^c$.

4. $(U\widehat{Q}V)_{i,j}$ is the probability that upon returning to E the first state visited is $j \in E$, given that the process started in state $i \in E$.

We now set up the required notation in order to define the R and G-measures. We begin with an arbitrary block-partitioned transition matrix, stochastic or strictly substochastic:

$$P = \left[\begin{array}{ccccc} P_{0,0} & P_{0,1} & P_{0,2} & \cdots & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & \cdots & \cdots \\ P_{2,0} & P_{2,1} & P_{2,2} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right], \tag{4}$$

where $P_{i,i}$, for $i = 0, 1, \ldots$, is a matrix of size $m_i \times m_i$. Here, the state space S has been partitioned as

$$S = \bigcup_{i=0}^{\infty} L_i \tag{5}$$

with

$$L_i = \{(i,1),(i,2),\ldots,(i,m_i)\}. \tag{6}$$

For the state (i,r), i is called the level variable and r the stage or phase variable. We also use the notation

$$L_{\le i} = \bigcup_{k=0}^{i} L_k, \tag{7}$$

for the set of all states in levels up to i, and $L_{\ge i}$ denotes the complement of $L_{\le(i-1)}$. We also note that, in what follows, if $E = L_{\le n}$, then the censored transition matrix, P^E, will be denoted by $P^{[\le n]}$. If $E = L_0$, then P^E is denoted by $P^{[0]}$.

For $0 \le i \le j$, $R_{i,j}$ is an $m_i \times m_j$ matrix whose (r,s)th entry is the expected number of visits to state (j,s) before hitting any state in $L_{\le(j-1)}$, given that the process starts in state (i,r). For $i > j \ge 0$, $G_{i,j}$ is an $m_i \times m_j$ matrix whose (r,s)th entry is the probability of hitting state (j,s) when the process enters $L_{\le(i-1)}$ for the first time, given that the process starts in state (i,r). It should be emphasized that the R-measure is defined for $i=j$, but the G-measure is not.

Using the probabilistic interpretations 2. and 3. above, we can establish useful relations between the R and G measures and the fundamental matrix $\hat{Q}$. To accomplish this, we re-partition the matrix P according to $L_{\le(n-1)}$, L_n and $L_{\ge(n+1)}$:

$$P = \begin{bmatrix} T & U_0 & U_1 \\ V_0 & Q_0 & U_2 \\ V_1 & V_2 & Q_1 \end{bmatrix}. \tag{8}$$

Let

$$Q = \begin{bmatrix} Q_0 & U_2 \\ V_2 & Q_1 \end{bmatrix},$$

let $\hat{Q}$ be partitioned accordingly:

$$\hat{Q} = \begin{bmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{bmatrix},$$

and let $R_{<n} = (R_{0,n}, R_{1,n}, \ldots, R_{n-1,n})^t$, where the superscript t stands for the transpose of the matrix. From the second probabilistic interpretation above, it is clear that

$$R_{<n} = (U_0, U_1)\begin{pmatrix} H_{1,1} \\ H_{2,1} \end{pmatrix}, \tag{9}$$

and using an obvious extension,

$$R_{n,n} = (Q_0, U_2)\begin{pmatrix} H_{1,1} \\ H_{2,1} \end{pmatrix}. \tag{10}$$

Similarly, if we let $G_{<n} = (G_{n,0}, G_{n,1}, \ldots, G_{n,n-1})$, then the third probabilistic interpretation can be applied to obtain

$$G_{<n} = (H_{1,1}, H_{1,2}) \begin{pmatrix} V_0 \\ V_1 \end{pmatrix}. \tag{11}$$

In the block-repeating case (1), the matrices $R_{i,j}$ and $G_{i,j}$ only depend on the value of $|i-j|$ except for $R_{0,j}$ and $G_{i,0}$ (for example, Grassmann and Heyman, 1990 or Zhao, Li and Braun, 1998). We then define

$$R_k = R_{i,j}, \quad \text{for } k = 0, 1, \ldots, \text{ with } k = j - i \text{ and } j \geq i > 0 \tag{12}$$

and

$$G_k = G_{i,j}, \quad \text{for } k = 1, 2, \ldots, \text{ with } k = i - j \text{ and } i > j > 0. \tag{13}$$

All R_i and G_i have a common size $m \times m$.

3 Invariance of Measures Under Censoring

As mentioned in the Introduction, the censoring technique will be used to attain a unified treatment of transition matrices with block-repeating entries. In this section, we provide some properties of censoring that move us toward this goal. We prove that both the R and G-measures are invariant under censoring. This invariance property plays an important role in derivations dealing with matrices with block-repeating property.

We also show that two other sequences of matrices, the A and B-measures, are invariant under censoring. These measures have probabilistic significance; for example, the stationary probability vector π for a positive recurrent Markov chain or the generalized stationary vector for a null recurrent Markov chain is essentially equivalent to the A-measure. The section concludes with a result on the invariance of the fundamental matrix under censoring.

We begin by collecting a number of useful basic properties of the censored Markov chain.

Lemma 1 *Let P be the transition matrix of a Markov chain, which is possibly strictly substochastic, and let E be a subset of the state space. Then,*

i) *P is irreducible if and only if P^E is irreducible for all E.*

ii) *P is recurrent if and only if P^E is recurrent for all E.*

iii) *P is transient if and only if P^E is transient for all E.*

iv) *if P is irreducible, then P is recurrent if and only if P^E is recurrent for some E.*

v) *if P is irreducible, then P is transient if and only if P^E is transient for some E.*

vi) *for $E_1 \subseteq E_2$, $P^{E_1} = \left(P^{E_2}\right)^{E_1}$.*

This lemma can be proved easily using the sample path structure of the censored Markov chain, and we omit the details. It is also possible to provide proofs of the invariance properties given later using sample path arguments. However, we prefer to use the expression for the fundamental matrix in partitioned form given in the next lemma (Lemma 2), since it leads to proofs which tie in with the probabilistic interpretations developed earlier. This lemma is also very useful in developing recursive expressions for the R and G-measures; such expressions have potential to be developed into computational schemes for these measures.

Lemma 2 *Let Q be a transition matrix with state space S and all states transient. Let E be any non-empty subset of S and let Q be partitioned according to E and its complement E^c:*

$$Q = \begin{bmatrix} Q_0 & U \\ V & Q_1 \end{bmatrix}.$$

Then, the fundamental matrix $\hat{Q} = \sum_{i=0}^{\infty} Q^i$ is given by

$$\hat{Q} = \begin{bmatrix} (I - Q_0 - U\hat{Q}_1 V)^{-1} & (I - Q_0 - U\hat{Q}_1 V)^{-1} U\hat{Q}_1 \\ \hat{Q}_1 V (I - Q_0 - U\hat{Q}_1 V)^{-1} & \hat{Q}_1 + \hat{Q}_1 V (I - Q_0 - U\hat{Q}_1 V)^{-1} U\hat{Q}_1 \end{bmatrix}. \tag{14}$$

where I is an identity matrix and $(I - X)^{-1} = \sum_{i=0}^{\infty} X^i$ is the minimal non-negative inverse of $I - X$ if the inverse is not unique.

Proof: Let $\hat{Q}$ be partitioned according to E and E^c as

$$\hat{Q} = \begin{bmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{bmatrix}.$$

$\hat{Q}$ is the minimal non-negative solution for X of the matrix equation $(I - Q)X = I$ (Proposition 5-11 of Kemeny, Snell and Knapp, 1976); or

$$(I - Q_0)X_{1,1} - UX_{2,1} = I, \tag{15}$$

$$-VX_{1,1} + (I - Q_1)X_{2,1} = 0, \tag{16}$$

$$(I - Q_0)X_{1,2} - UX_{2,2} = 0, \tag{17}$$

$$-VX_{1,2} + (I - Q_1)X_{2,2} = I, \tag{18}$$

when X is partitioned according to E and E^c into

$$X = \begin{bmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{bmatrix}.$$

Equation (16) is equivalent to $(I - Q_1)X_{2,1} = VX_{1,1}$. It follows from Proposition 5-11 of Kemeny, Snell and Knapp (1976) that

$$X_{2,1} = (I - Q_1)^{-1} V X_{1,1} \tag{19}$$

is the minimal non-negative solution of (16) for a fixed $X_{1,1}$, where $(I-Q_1)^{-1}=\sum_{i=0}^{\infty} Q_1^i$ is the minimal non-negative inverse of $I-Q_1$. Therefore, $H_{2,1}=(I-Q_1)^{-1}VX_{1,1}$ if $X_{1,1}$ is the minimal non-negative solution of (15). Using (19) in (15), we have $(I-Q_0)X_{1,1}-U(I-Q_1)^{-1}VX_{1,1}=I$ or $\{I-[Q_0+U(I-Q_1)^{-1}V]\}X_{1,1}=I$. This implies that

$$H_{1,1}=[I-Q_0-U(I-Q_1)^{-1}V]^{-1}$$

and hence

$$H_{2,1}=(I-Q_1)^{-1}V[I-Q_0-U(I-Q_1)^{-1}V]^{-1}.$$

We can similarly prove that

$$H_{1,2}=[I-Q_0-U(I-Q_1)^{-1}V]^{-1}U(I-Q_1)^{-1}$$

and

$$H_{2,2}=(I-Q_1)^{-1}\{I+V[I-Q_0-U(I-Q_1)^{-1}V]^{-1}U(I-Q_1)^{-1}\}.$$

■

For convenience, we put a symmetric expression for $\hat{Q}$ in the following corollary.

Corollary 3 *The fundamental matrix $\hat{Q}=\sum_{i=0}^{\infty} Q^i$ can be also expressed by*

$$\hat{Q}=\left[\begin{array}{cc} \hat{Q}_0+\hat{Q}_0U(I-Q_1-V\hat{Q}_0U)^{-1}V\hat{Q}_0 & \hat{Q}_0U(I-Q_1-V\hat{Q}_0U)^{-1} \\ (I-Q_1-V\hat{Q}_0U)^{-1}V\hat{Q}_0 & (I-Q_1-V\hat{Q}_0U)^{-1} \end{array}\right]. \tag{20}$$

Remark 1 *The matrix Q in the above lemma could be either stochastic or strictly substochastic. The expressions in (14) and (20) are well known if the size of the matrices involved are all finite.*

We are now ready to demonstrate that the R and G-measures are invariant under censoring.

Theorem 4 *For an arbitrary Markov chain whose transition matrix P is partitioned according to levels as in (4), let $R_{i,j}$ and $G_{i,j}$ be the R and G-measures, respectively, defined for the Markov chain P, and let $R_{i,j}^{[\leq n]}$ and $G_{i,j}^{[\leq n]}$ be the R and G-measures, respectively, defined for the censored Markov chain with censoring set $L_{\leq n}$. For given $0=i<j$ or $1\leq i\leq j$,*

$$R_{i,j}^{[\leq n]}=R_{i,j} \tag{21}$$

for all $n\geq j$; and for given $0\leq j<i$,

$$G_{i,j}^{[\leq n]}=G_{i,j} \tag{22}$$

for all $n\geq i$.

Proof: We only prove the first result; the second can be proved similarly.

First, assume $n = j$, and let P be re-partitioned according to $L_{\le(n-1)}$, L_n and $L_{\ge(n+1)}$ as in (8), and define Q, $\hat{Q}$, and $R_{<n}$ as in the displays subsequent to (8). Applying (9) together with (14), it becomes clear that

$$R_{<n} = (U_0, U_1) \begin{pmatrix} (I - Q_0 - U\hat{Q}_1 V)^{-1} \\ \hat{Q}_1 V (I - Q_0 - U\hat{Q}_1 V)^{-1} \end{pmatrix},$$

and similarly, using (10) with (14), we have

$$R_{n,n} = (Q_0, U_2) \begin{pmatrix} (I - Q_0 - U\hat{Q}_1 V)^{-1} \\ \hat{Q}_1 V (I - Q_0 - U\hat{Q}_1 V)^{-1} \end{pmatrix},$$

On the other hand, the censored matrix with censoring set $L_{\le n}$ is given by

$$P^{[\le n]} = \begin{bmatrix} T + U_1\hat{Q}_1 V_1 & U_0 + U_1\hat{Q}_1 V_2 \\ V_0 + U_2\hat{Q}_1 V_1 & Q_0 + U_2\hat{Q}_1 V_2 \end{bmatrix}.$$

Therefore,

$$R^{[\le n]}_{<n} = (U_0 + U_1\hat{Q}_1 V_2)(I - Q_0 - U_2\hat{Q}_1 V_2)^{-1},$$

where $R^{[\le n]}_{<n} = (R^{[\le n]}_{0,n}, R^{[\le n]}_{1,n}, \ldots, R^{[\le n]}_{n-1,n})^t$. Similarly,

$$R^{[\le n]}_{n,n} = (Q_0 + U_2\hat{Q}_1 V_2)(I - Q_0 - U_2\hat{Q}_1 V_2)^{-1}.$$

Now, $R_{i,n} = R^{[\le n]}_{i,n}$ for all $i \le n$.

The above result, together with vi) of Lemma 1, implies that (21) is also valid for $n > j$. In fact, if $n > j$, we censor the matrix P first using the censoring set $L_{\le j}$ and we know $R_{i,j} = R^{[\le j]}_{i,j}$. Next, censor the matrix P using the censoring set $L_{\le n}$. It follows from vi) of Lemma 1 that the censored matrix $P^{[\le j]}$ can be obtained by censoring $P^{[\le n]}$ using the censoring set $L_{\le j}$ and therefore $R^{[\le n]}_{i,j} = R^{[\le j]}_{i,j}$, the fact just proved above (using $P^{[\le n]}$ in place of P). ■

Remark 2 *That the R and G-measures are invariant under censoring was observed by Grassmann and Heyman (1990); they also provided a proof to a special case of the above theorem.*

As we mentioned earlier, one of the advantages of introducing the R and G-measures is that they can be used to express other interesting measures.

For an arbitrary Markov chain defined by (4), we define the A and B-measures as follows. For $i \ge 0$ and $j \ge 0$ with $i \ne j$, define $A_{i,j}$ to be a matrix of size $m_i \times m_j$ whose (r,s)th entry is the expected number of visits to state (j,s) before hitting any state in level i, given that the process starts in state (i,r). For $i \ge 0$ and $j \ge 0$, define $B_{i,j}$ to be a matrix of size $m_i \times m_j$. When $i \ne j$, the (r,s)th entry of $B_{i,j}$ is the probability of visiting state (j,s) for the first time before hitting any state in level j, given that the process starts in state (i, r). When $i = j$, the (r, s)th entry of $B_{i,j}$ is the probability of returning to level j for the first time by hitting state (j, s), given that the process starts in state (i, r). For the purpose of this paper, it suffices to consider $A_{i,j}$ for a fixed i and $B_{i,j}$ for a fixed j only. Without loss of generality, we consider only $A_j = A_{0,j}$ and $B_i = B_{i,0}$.

To begin to see why the A and B-measures are important, consider the following. In the scalar case, the stationary probability vector π satisfies $\pi = c(1, A_1, A_2, \ldots)$, where c is a normalization constant, when the chain is positive recurrent. In fact, for a recurrent chain, $(1, A_1, A_2, \ldots)$ is the unique solution, up to multiplication by a constant, of the stationary equations $x = xP$. In the block case, let the stationary probability vector be partitioned according to levels: $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$. Then, $\pi = \pi_0(I, A_1, A_2, \ldots)$, where π_0 is the unique solution, up to multiplication by a constant, of $\pi_0 = \pi_0 P^{[0]}$, where I is an identity matrix and $P^{[0]}$ is the censored matrix to level 0. π_0 can be viewed as the vector-valued normalization constant. The B-measure is a dual of the A-measure.

The following recursive formulas for computing the A and B-measures in terms of the R and G-measures were derived by Zhao, Li and Braun (1998) when all the block entries have a common size. Their proof can be easily extended to the case of different block sizes.

Lemma 5 *Matrices A_n and $R_{k,n}$ satisfy*

$$A_n = \begin{cases} R_{0,1}, & \text{if } n = 1, \\ R_{0,n} + \sum_{k=1}^{n-1} A_k R_{k,n}, & \text{if } n \geq 2, \end{cases} \tag{23}$$

and matrices B_n and $G_{n,k}$ satisfy

$$B_n = \begin{cases} G_{1,0}, & \text{if } n = 1, \\ G_{n,0} + \sum_{k=1}^{n-1} G_{n,k} B_k, & \text{if } n \geq 2. \end{cases} \tag{24}$$

As a consequence of Theorem 4 and the above lemma, the A and B-measures are invariant under censoring, as stated in the following corollary.

Corollary 6 *Let A_j and B_i be the respective A and B-measures for a Markov chain having a transition matrix of the form (4), and let $A_j^{[\leq n]}$ and $B_i^{[\leq n]}$ be the A and B-measures, respectively, defined for the censored Markov chain with censoring set $L_{\leq n}$. For given i and j,*

$$A_j^{[\leq n]} = A_j \tag{25}$$

for all $n \geq j$ and

$$B_i^{[\leq n]} = B_i \tag{26}$$

for all $n \geq i$.

When the transition matrix P has the property of repeating blocks as in (1) the result in Lemma 5 can be simplified as follows.

Corollary 7 *For the Markov chain whose transition matrix is given as in (1), matrices A_n and B_n satisfy*

$$A_n = \begin{cases} R_{0,1}, & \textit{if } n = 1, \\ R_{0,n} + \sum_{k=1}^{n-1} A_{n-k} R_k, & \textit{if } n \geq 2, \end{cases} \tag{27}$$

and

$$B_n = \begin{cases} G_{1,0}, & \textit{if } n = 1, \\ G_{n,0} + \sum_{k=1}^{n-1} G_k B_{n-k}, & \textit{if } n \geq 2, \end{cases} \tag{28}$$

where R_k and G_k are defined as in (12) and (2).

It follows from the relations in (9), (10) and (11) that the fundamental matrix $\hat{Q}$ contains essential information about the R and G-measures, and therefore about the A and B-measures. Thus, the fundamental matrix merits study in its own right; this we do in the next section, but we first conclude this section by showing that the values in a fundamental matrix are invariant under censoring.

Theorem 8 *Let Q be a stochastic or strictly substochastic transition matrix with state space S and all states transient, and let $q_{i,j}$ be the (i,j)th entry of the fundamental matrix $\hat{Q} = \sum_{k=0}^{\infty} Q^k$. Let E be any subset of S containing states i and j, and let Q^E be the censored matrix with censoring set E. Then, the entry corresponding to states i and j of the fundamental matrix $\widehat{Q^E} = \sum_{k=0}^{\infty} (Q^E)^k$ for the censored matrix Q^E is equal to $q_{i,j}$.*

Proof: Partition Q according to E and the complement E^c of E:

$$Q = \begin{bmatrix} Q_0 & U \\ V & Q_1 \end{bmatrix}.$$

Then, $Q^E = Q_0 + U\hat{Q}_1 V$ and $\widehat{Q^E} = (I - Q_0 - U\hat{Q}_1 V)^{-1}$, which is equal to the block corresponding to E in the fundamental matrix $\hat{Q}$ according to Lemma 2. ■

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